Unit 3

COORDINATE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *understand specific facts and principles about lines and circles.*
- *know basic concepts about conic sections.*
- *know methods and procedures for solving problems on conic sections.*

Main Contents

3.1 STRAIGHT LINE

3.2 CONIC SECTIONS

Key terms

Summary

Review Exercises

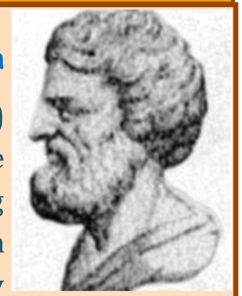
INTRODUCTION

THE METHOD OF ANALYTIC GEOMETRY REDUCES A PROBLEM IN GEOMETRY TO AN AI
PROBLEM BY ESTABLISHING A CORRESPONDENCE BETWEEN A CURVE AND A DEFINITE EQUA

THE CONCEPTS OF LINES AND CONICS OCCUR IN NATURE AND ARE USED IN MANY PH
SITUATIONS IN NATURE, ENGINEERING AND SCIENCE. FOR INSTANCE, THE EARTH'S ORBIT A
SUN IS ELLIPTICAL, WHILE MOST SATELLITE DISHES ARE PARABOLIC.

IN THIS UNIT, YOU WILL STUDY SOME MORE ABOUT STRAIGHT LINES AND CIRCLES, AND
PROPERTIES OF THE CONIC SECTIONS: *circle, parabola, ellipse* AND *hyperbola*.

HISTORICAL NOTE



Apollonius of Perga

The Greek mathematician Apollonius (who died about 200 B.C.) studied conic sections. Apollonius is credited with providing the names "ellipse", "parabola", and "hyperbola" and for discovering that all the conic sections result from intersection of a cone and a plane. The theory was further advanced to its fullest form by Fermat, Descartes and Pascal during the 17th century.

OPENING PROBLEM

A PARABOLIC ARCH HAS DIMENSIONS A
THE FIGURE. CAN YOU FIND THE EQUA
PARABOLA? WHAT ARE THE RESPECTIV y
FOR x = 5, 10 AND 15?

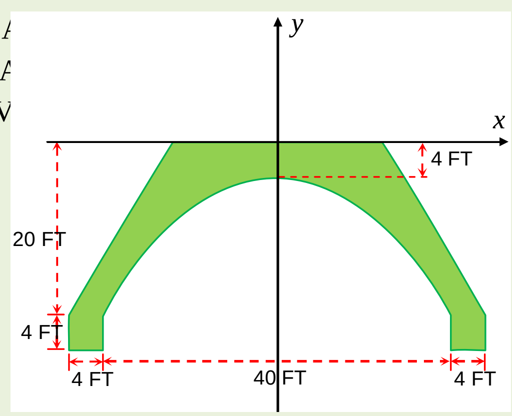


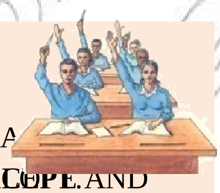
Figure 3.1

3.1 STRAIGHT LINE

Review on equation of a straight line

IN GRADE 10, YOU HAVE LEARNT HOW TO FIND THE EQUATION OF A LINE AND HOW TO TELL W
TWO LINES ARE PARALLEL OR PERPENDICULAR BY LOOKING AT THEIR SLOPES. NOW LET US I
CONCEPTS WITH THE FOLLOWING ACTIVITY.

ACTIVITY 3.1



1. GIVEN TWO POINTS P (1, 4) AND Q (3, –2), FIND THE EQUATION OF A LINE PASSING THROUGH P AND Q; AND IDENTIFY ITS SLOPE AND y-INTERCEPT.
2. GIVEN THE FOLLOWING EQUATIONS OF LINES, CHARACTERIZE EACH AS VERTICAL, HORIZONTAL OR NEITHER.

   A $y = 3x - 5$ **B** $y = 7$ **C** $x = 2$ **D** $x + y = 0$

3. IDENTIFY EACH OF THE FOLLOWING PAIRS OF LINES AS PARALLEL, PERPENDICULAR OR INTERSECTING (BUT NOT PERPENDICULAR).

   A $\ell_1 : y = 2x + 3;\ \ell_2 : y = \frac{1}{2}x - 2$

   B $\ell_1 : y = 2x + 3;\ \ell_2 : y = -\frac{1}{2}x - 3$

   C $\ell_1 : y = 2x + 3;\ \ell_2 : y = 2x + 5$

   D $\ell_1 : 3x + 4y - 8 = 0\ \ \ell_2 : 4x - 3y - 9 = 0$

FROM THE ABOVE ACTIVITY, YOU CAN SUMMARIZE AS FOLLOWS.

- ✓ ANY TWO POINTS DETERMINE A STRAIGHT LINE.
- ✓ IF P(x_1, y_1) AND Q(x_2, y_2) ARE POINTS ON A LINE WITH $x_1 \neq x_2$, THEN

  $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$ IS THE EQUATION OF THE STRAIGHT LINE AND THE RATIO

  $m = \frac{y_2 - y_1}{x_2 - x_1}$ IS THE SLOPE OF THE LINE.

- ✓ IF $x_2 = x_1$, THEN THE LINE IS VERTICAL AND ITS EQUATION IS $x = x_1$. IN THIS CASE THE LINE HAS NO SLOPE.
- ✓ IF TWO LINES ℓ_1 AND ℓ_2 HAVE THE SAME SLOPE, THEN THE TWO LINES ARE PARALLEL.

- ✓ IF THE PRODUCT OF THE SLOPES OF ℓ_1 AND ℓ_2 IS −1, THEN THE TWO LINES ARE PERPENDICULAR.
- ✓ IF THE EQUATION OF A LINE IS GIVEN BY $y = mx + b$, THEN m IS THE SLOPE OF THE LINE AND b IS ITS y - INTERCEPT.

Example 1 FIND THE EQUATION OF THE LINE THAT PASSES THROUGH THE POINTS (−3, 2) AND (4, 7) AND IDENTIFY ITS SLOPE.

Solution THE SLOPE IS GIVEN BY $m = \dfrac{7-2}{4-(-3)} = \dfrac{5}{7}$

THUS, FOR ANY POINT P(x, y) ON THE LINE, $\dfrac{y-2}{x-(-3)} = \dfrac{5}{7} \Leftrightarrow y = \dfrac{5}{7}x + \dfrac{29}{7}$

3.1.1 Angle Between Two Lines on the Coordinate Plane

IN THE PREVIOUS SECTION, YOU HAVE SEEN HOW TO IDENTIFY WHETHER TWO LINES ARE PARALLEL OR PERPENDICULAR. NOW, WHEN TWO LINES ARE INTERSECTING, YOU WILL SEE HOW TO DE ANGLE BETWEEN THE TWO LINES AND HOW TO DETERMINE THIS ANGLE.

Group Work 3.1

CONSIDER THE FOLLOWING GRAPH AND ANSWER THE QU… LLOW:

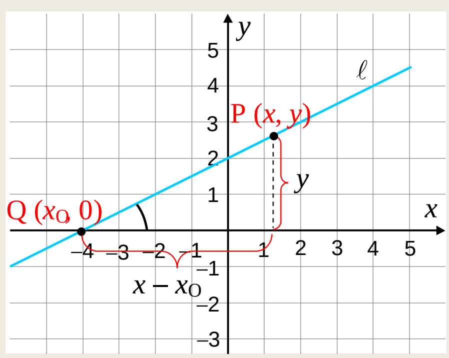


Figure 3.2

FIND

A TAN α

B SLOPE OF THE LINE ℓ

C THE RELATION BETWEEN TAN α AND SLOPE OF ℓ

D IF ℓ IS VERTICAL, THEN α = ___.

E IF ℓ IS HORIZONTAL, THEN α = ____.

F IF $\alpha > 90°$, DO YOU GET THE SAME RELATION AS IN **C** ABOVE BETWEEN α AND THE SLOPE OF THE LINE ℓ?

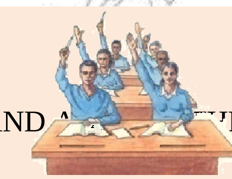
Definition 3.1

THE ANGLE α MEASURED FROM THE POSITIVE x-AXIS TO A LINE IN THE COUNTER-CLOCKWISE DIRECTION IS CALLED THE **angle of inclination** OF THE LINE.

Example 2 IF THE ANGLE OF INCLINATION OF A LINE IS 120°, THEN ITS SLOPE IS TAN 120° = $-\sqrt{3}$.

Example 3 IF THE SLOPE OF A LINE IS 1, THEN ITS ANGLE OF INCLINATION IS 45°.

ACTIVITY 3.2



CONSIDER THE FOLLOWING TWO INTERSECTING LINES, AND ANSWER THE QUESTIONS THAT FOLLOW:

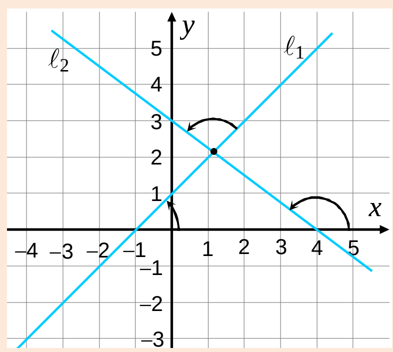


Figure 3.3

A WHAT IS THE ANGLE OF INCLINATION OF ℓ_1?

B WHAT IS THE ANGLE OF INCLINATION OF ℓ_2?

C CAN YOU FIND ANY RELATION BETWEEN α, β AND θ?

Definition 3.2

THE **angle between two intersecting lines** ℓ_1 AND ℓ_2 IS DEFINED TO BE THE ANGLE θ MEASURED COUNTER-CLOCKWISE FROM ℓ_1 TO ℓ_2.

FROM THE ABOVE ACTIVITY, YOU HAVE = – , SLOPE OF ℓ_1 = TAN AND SLOPE OF ℓ_2 = TAN

THUS = – $\Rightarrow$ TAN = TAN (–) = $\dfrac{\text{TAN} - \text{TAN}}{1 + \text{TAN}\ \text{TAN}}$

HENCE IF m_1 IS THE SLOPE OF ℓ_1 AND m_2 IS THE SLOPE OF ℓ_2, THEN THE TANGENT OF THE ANGLE BETWEEN TWO LINES ℓ_1 AND ℓ_2 MEASURED FROM ℓ_1 TO ℓ_2 COUNTER-CLOCKWISE IS GIVEN BY

$$\text{TAN} = \frac{m_2 - m_1}{1 + m_1 m_2},\ \text{IF } m_1 m_2 \neq -1.$$

SO, THE ANGLE CAN BE FOUND FROM THE ABOVE EQUATION.

Note:

THE DENOMINATOR $1 + m_1 m_2 = 0 \Leftrightarrow m_1 m_2 = -1 \Leftrightarrow$ TAN IS UNDEFINED $\Leftrightarrow$ $= 90^{O}$.

THUS, THE ANGLE BETWEEN THE TWO LINES IS 90^{O} $\Leftrightarrow m_1 m_2 = -1$ OR $m_1 = -\dfrac{1}{m_2}$

Example 4 GIVEN POINTS P(2, 3), Q(–4, 1), C(2, 4) AND D(6, 5), FIND THE TANGENT OF THE ANGLE BETWEEN THE LINE THAT PASSES THROUGH P AND Q AND THE LINE THAT PASSES THROUGH C AND D WHEN MEASURED FROM THE LINE THAT PASSES THROUGH P AND Q TO THE LINE THAT PASSES THROUGH C AND D COUNTER-CLOCKWISE.

Solution LET m_1 BE THE SLOPE OF THE LINE THROUGH P AND Q AND m_2 BE THE SLOPE OF THE LINE THROUGH C AND D.

THEN, $m_1 = \dfrac{1 - 3}{-4 - 2} = \dfrac{-2}{-6} = \dfrac{1}{3}$ AND $m_2 = \dfrac{5 - 4}{6 - 2} = \dfrac{1}{4}$.

THUS, THE TANGENT OF THE ANGLE BETWEEN THE LINE THROUGH P AND Q AND THE LINE THROUGH C AND D IS

$$\text{TAN} = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{1}{4} - \frac{1}{3}}{1 + \frac{1}{3}\cdot\frac{1}{4}} = \frac{\frac{3-4}{12}}{\frac{12+1}{12}} = \frac{-1}{13}$$

Exercise 3.1

1 WRITE DOWN THE EQUATION OF THE LINE THAT:

A PASSES THROUGH (–6, 2) AND HAS SLOPE $m = 4$

B PASSES THROUGH (6, 6) AND (–1, 7)

C PASSES THROUGH (2, – 4) AND IS PARALLEL TO THE LINE WITH EQUATION $y = 7x - 10$.

D PASSES THROUGH (2, –4) AND IS PERPENDICULAR TO THE LINE WITH EQUATION $y = 2x - 1$.

E PASSES THROUGH (1, 3) AND THE ANGLE FROM THE LINE WITH EQUATION $y = x + 2$ TO THE LINE IS 45°.

2 FIND THE TANGENT OF THE ANGLE BETWEEN THE GIVEN LINES.

A ℓ_1: $y = -3x + 2$; ℓ_2: $y = -x$ **B** ℓ_1 : $3x - y - 2 = 0$; ℓ_2: $4x - y - 6 = 0$

3 DETERMINE B SO THAT THE LINE WITH EQUATION $x + By - 6 = 0$ IS:

A PARALLEL TO THE LINE WITH EQUATION $y = \frac{4}{7}x + 1$

B PERPENDICULAR TO THE LINE WITH EQUATION $y = \frac{4}{7}x + 1$

4 A CAR RENTAL COMPANY LEASES AUTOMOBILES FOR A CHARGE OF 20 BIRR/DAY PLUS 2 BIRR/KM. WRITE AN EQUATION FOR THE COST y IN TERMS OF THE DISTANCE x DRIVEN, IF THE CAR IS LEASED FOR 5 DAYS.

5 WATER IN A LAKE WAS POLLUTED WITH SEWAGE FROM A FACTORY AT A LEVEL OF 47 *mg* OF WASTE COMPOUNDS PER 1000 *l* OF WATER. IT IS DETERMINED THAT THE POLLUTION LEVEL WOULD DROP AT THE RATE OF 0.75 *mg* OF WASTE COMPOUNDS PER 1000 ℓ OF WATER PER YEAR, IF A PLAN PROPOSED BY ENVIRONMENTALISTS IS FOLLOWED. LET $x = 0$ CORRESPOND TO 2001 AND SUCCESSIVE YEARS CORRESPOND TO $x = 1, 2, 3, \ldots$. FIND THE EQUATION $y = mx + b$ THAT HELPS PREDICT THE POLLUTION LEVEL IN FUTURE YEARS, IF THE PLAN IS IMPLEMENTED.

3.1.2 Distance between a Point and a Line on the Coordinate Plane

ACTIVITY 3.3

GIVEN A LINE ℓ AND A POINT P NOT ON ℓ,

A DRAW LINE SEGMENTS FROM P TO POINTS ON ℓ (AS MANY AS POSSIBLE).

B WHICH LINE SEGMENT HAS THE SHORTEST LENGTH?

Definition 3.3

SUPPOSE A LINE ℓ AND A POINT P(x, y) ARE GIVEN. IF P DOES NOT LIE ON ℓ, THEN WE DEFINE THE DISTANCE d FROM P TO ℓ AS THE PERPENDICULAR DISTANCE BETWEEN P AND ℓ. IF P IS ON ℓ, THE DISTANCE IS TAKEN TO BE ZERO.

LET A LINE ℓ: $Ax + By + C = 0$ WITH A, B AND C ALL NON-ZERO BE GIVEN. TO FIND THE DISTANCE FROM THE ORIGIN TO THE LINE $Ax + By + C = 0$, YOU CAN DO THE FOLLOWING:

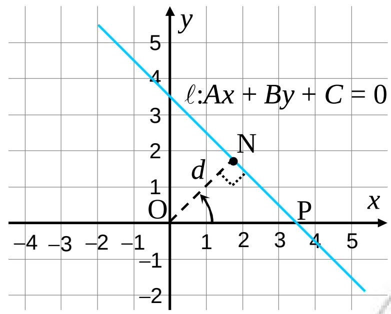


Figure 3.4

DRAW $\overline{ON}$ PERPENDICULAR TO $Ax + By + C = 0$. ΔONP IS RIGHT ANGLED TRIANGLE. THUS

$$\left|\text{COS}\ \right| = \frac{d}{\text{OP}} \Rightarrow d = \text{OP}\left|\text{COS}\ \right|.$$

THE x-INTERCEPT OF $Ax + By + C = 0$ IS $-\frac{C}{A}$.

THUS, $d = \frac{|C|}{|A|}\left|\text{COS}\ \right|$

AGAIN $\overline{ON}$ BEING $\perp$ TO THE LINE $Ax + By + C = 0$ GIVES: SLOPE OF $\overline{ON}$ = TAN $= \frac{B}{A}$

$\left(\text{BECAUSE SLOPE OF } Ax + By + C = 0 \text{ IS } \frac{-A}{B}\right)$

THIS GIVES $\left|\text{COS}\ \right| = \frac{|A|}{\sqrt{A^2 + B^2}}$

HENCE, THE DISTANCE FROM THE ORIGIN TO ANY LINE $Ax + By + C = 0$ WITH $A \neq 0$, $B \neq 0$ AND $C \neq 0$ IS GIVEN BY $\frac{|C|}{\sqrt{A^2 + B^2}}$

Note:

THE ABOVE FORMULA IS TRUE WHEN

I $C = 0$ (*in this case you get a line through the origin*) OR

II EITHER $A = 0$ OR $B = 0$ BUT NOT BOTH, WITH $C \neq 0$ ($A = 0$ AND $B \neq 0$ GIVES A HORIZONTAL LINE, WHILE $A \neq 0$ AND $B = 0$ GIVES A VERTICAL LINE).

Example 5 FIND THE DISTANCE FROM THE ORIGIN TO THE LINE $5x - 2y - 7 = 0$.

Solution THE DISTANCE $= \dfrac{|-7|}{\sqrt{5^2 + (-2)^2}} = \dfrac{7}{\sqrt{29}}$

Group Work 3.2

1 CONSIDER A POINT P (h, k) ON THE xy-COORDINATE SYSTEM. A NEW $x'y'$ COORDINATE SYSTEM SUCH THAT

A THE ORIGIN OF THE NEW SYSTEM IS AT P (h, k)

B THE x'-AXIS IS PARALLEL TO THE x-AXIS AND THE y'-AXIS IS PARALLEL TO THE y-AXIS.

LET P BE A POINT ON THE PLANE SUCH THAT IT HAS COORDINATES (x, y) IN THE xy-SYSTEM AND P(x', y') IN THE $x'y'$-SYSTEM. EXPRESS x' AND y' IN TERMS OF x, y, h AND k.

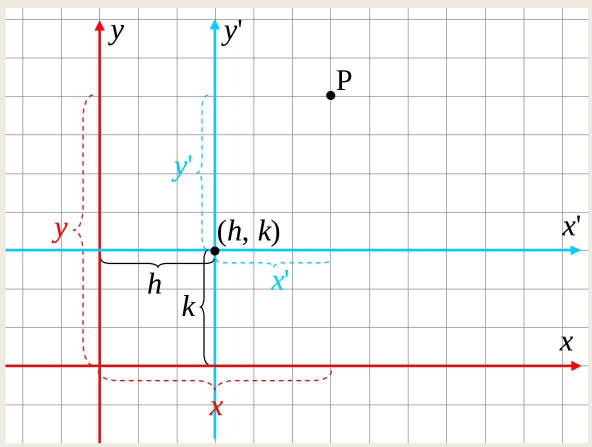


Figure 3.5

2 IF $(h, k) = (3, 4)$, WHAT IS THE REPRESENTATION OF P(–3, 2) (GIVEN IN THE xy-SYSTEM) IN THE NEW $x'y'$-SYSTEM?

FROM THE ABOVE GROUP WORK, YOU SHOULD GET THE **translation formulas:**

$x' = x - h$

$y' = y - k$

WHERE (h, k) REPRESENTS THE ORIGIN OF THE NEW $x'y'$-SYSTEM AND (x', y') AND (x, y) REPRESENT THE COORDINATES OF A POINT IN THE $x'y'$ AND xy SYSTEMS, RESPECTIVELY.

Example 6 FIND THE NEW COORDINATES OF P(5, –3), IF THE AXES ARE TRANSLATED TO A NEW ORIGIN (–2, 3).

Solution THE FORMULAE ARE $x' = x - h$ AND $y' = y - k$. HERE, $(h, k) = (-2, -3)$

THUS, THE NEW COORDINATES OF $P(5, -3)$ ARE $x' = 5 - (-2) = 7$ AND $y' = -3 - (-3) = 0$

THUS, IN THE $x'y'$-SYSTEM, P(7, 0).

NEXT, WE WILL FIND THE DISTANCE BETWEEN ANY POINT $P(h, k)$ AND THE LINE

$\ell : Ax + By + C = 0.$

TRANSLATE THE COORDINATE SYSTEM TO A NEW ORIGIN AT $P(h, k)$.

LET THE EQUATION OF THE LINE IN THE NEW $x'y'$-SYSTEM BE $A'x' + B'y' + C' = 0$. THEN, THE DISTANCE FROM P TO ℓ IS GIVEN BY $\dfrac{|C'|}{\sqrt{A'^2 + B'^2}}$

BUT, $A'x' + B'y' + C' = 0 \Leftrightarrow A'(x - h) + B'(y - k) + C' = 0$

$$A'x - A'h + B'y - B'k + C' = 0$$

$$A'x + B'y + (C' - A'h - B'k) = 0$$

SINCE IN THE xy- SYSTEM THE EQUATION IS $Ax + By + C = 0$

YOU GET $A = A'$, $B = B'$, $C = C' - A'h - B'k$

SO, $C' = A'h + B'k + C = Ah + Bk + C$

HENCE THE DISTANCE FROM $P(h, k)$ TO ℓ IS GIVEN BY $\dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$

Example 7 FIND THE DISTANCE BETWEEN P(–4, 2) AND ℓ: $2x + 9y - 3 = 0$

Solution $d = \dfrac{|2(-4) + 9(2) - 3|}{\sqrt{2^2 + 9^2}} = \dfrac{|-8 + 18 - 3|}{\sqrt{85}} = \dfrac{7}{\sqrt{85}}$

Exercise 3.2

1 FIND THE DISTANCE OF EACH OF THE FOLLOWING LINES FROM THE ORIGIN.

A $4x - 3y = 10$ B $x - 5y + 2 = 0$

C $3x + y - 7 = 0$

2 FIND THE DISTANCE FROM EACH POINT TO THE GIVEN LINE

A P(–3, 2) ; $5x + 4y - 3 = 0$ B P(4, 0) ; $2x - 3y - 2 = 0$

C P(–3, –5); $2x - 3y + 11 = 0$

3.2 CONIC SECTIONS

3.2.1 Cone and Sections of a Cone

THE COORDINATE PLANE CAN BE CONSIDERED AS A SET OF POINTS WHICH CAN BE WRITTEN

$$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}.$$

IF SOME OF THE POINTS OF THE PLANE SATISFY A CERTAIN CONDITION, THEN THESE POINTS I SUBSET OF THE SET OF ALL POINTS (I.E. THE PLANE).

Definition 3.4

A **locus** IS A SYSTEM OF POINTS, LINES OR CURVES ON A PLANE WHICH SATISFY ONE OR M GIVEN CONDITIONS.

Example 1

THE FOLLOWING ARE EXAMPLES OF LOCI (PLURAL OF LOCUS).

1 THE SET $\{(x, y)\in \mathbb{R}^2 : y = 3x + 5\}$ IS A LINE IN THE COORDINATE PLANE.

2 THE SET OF ALL POINTS ON THE x-AXIS WHICH ARE AT A DISTANCE OF 3 UNITS FROM THE ORIGIN IS $\{(-3, 0), (3, 0)\}$.

IN THIS SUBSECTION, THE PLANE CURVES CALLED CIRCLES, PARABOLAS, ELLIPSES AND H WILL BE CONSIDERED.

CONSIDER TWO RIGHT CIRCULAR CONES WITH COMMON VERTEX AND WHOSE ALTITUDES LIE ON THE SAME LINE AS SHOWN IN FIGURE 3.6.

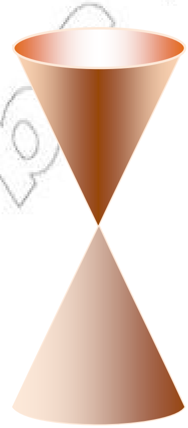
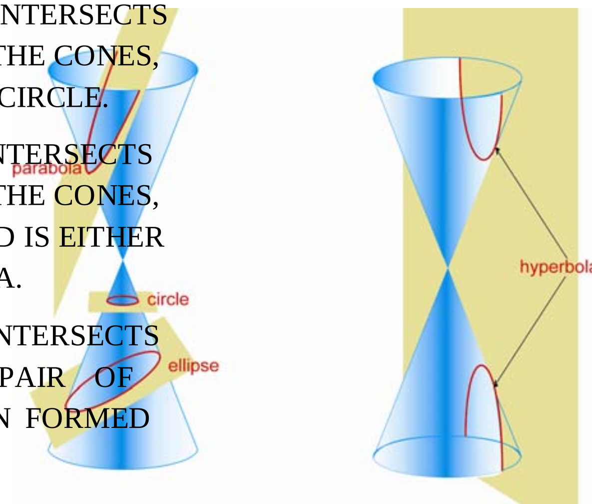


Figure 3.6



1. IF A HORIZONTAL PLANE INTERSECTS /SLICES THROUGH ONE OF THE CONES, THE SECTION FORMED IS A CIRCLE.
2. IF A SLANTED PLANE INTERSECTS /SLICES THROUGH ONE OF THE CONES, THEN THE SECTION FORMED IS EITHER AN ELLIPSE OR A PARABOLA.
3. IF A VERTICAL PLANE INTERSECTS /SLICES THROUGH THE PAIR OF CONES, THEN THE SECTION FORMED IS A HYPERBOLA.

Figure 3.7

SINCE EACH OF THESE PLANE CURVES ARE FORMED BY INTERSECTING A PAIR OF CONES WITH A PLANE, THEY ARE CALLED **conic sections**.

3.2.2 Circles

ACTIVITY 3.4

DESCRIBE EACH OF THE FOLLOWING LOCI.

- **A** THE SET OF ALL POINTS IN A PLANE WHICH ARE AT A DISTANCE OF 5 UNITS FROM THE ORIGIN.
- **B** THE SET OF ALL POINTS IN A PLANE WHICH ARE AT A DISTANCE OF 4 UNITS FROM THE POINT $P(1, -2)$.

EACH OF THE LOCI DESCRIBED IN ACTIVITY 3.4 REPRESENTS A CIRCLE.

Definition 3.5

A **circle** IS THE LOCUS OF A POINT THAT MOVES IN A PLANE WITH A FIXED DISTANCE FROM A FIXED POINT. THE **fixed distance** IS CALLED THE **radius** OF THE CIRCLE AND THE **fixed point** IS CALLED THE **centre** OF THE CIRCLE.

FROM THE ABOVE DEFINITION, FOR ANY POINT $P(x, y)$ ON A CIRCLE WITH CENTRE $C(h, k)$ AND RADIUS r, $PC = r$ AND BY THE DISTANCE FORMULA YOU HAVE,

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

FROM THIS, BY SQUARING BOTH SIDES, YOU GET

$$(x - h)^2 + (y - k)^2 = r^2$$

THE ABOVE EQUATION IS CALLED THE **standard form of the equation of a circle**, WITH CENTRE C(h, k) AND RADIUS r.

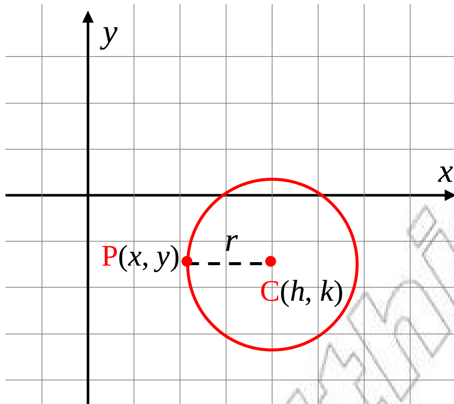


Figure 3.8

IF THE CENTRE OF A CIRCLE IS AT THE ORIGIN (I.E. $h = 0$, $k = 0$), THEN THE ABOVE EQUATION BECOMES,

$$x^2 + y^2 = r^2$$

THE ABOVE EQUATION IS CALLED THE **standard form of equation of a circle**, WITH CENTRE AT THE ORIGIN AND RADIUS r.

Example 2 WRITE DOWN THE STANDARD FORM OF THE EQUATION OF A CIRCLE WITH THE G CENTRE AND RADIUS.

A C (0, 0), $r = 8$ **B** C (2, –7), $r = 9$

Solution

A $h = k = 0$ AND $r = 8$

THEREFORE, THE EQUATION OF THE CIRCLE IS $(x - 0)^2 + (y - 0)^2 = 8^2$.

THAT IS, $x^2 + y^2 = 64$.

B $h = 2$, $k = -7$ AND $r = 9$.

THEREFORE, THE EQUATION OF THE CIRCLE IS $(x - 2)^2 + (y + 7)^2 = 9^2$.

THAT IS, $(x - 2)^2 + (y + 7)^2 = 81$.

Example 3 WRITE THE STANDARD FORM OF THE EQUATION OF THE CIRCLE WITH CENTRE AT AND THAT PASSES THROUGH THE POINT P (7, –3).

Solution LET r BE THE RADIUS OF THE CIRCLE. THEN THE EQUATION OF THE CIRCLE IS

$$(x-2)^2 + (y-3)^2 = r^2$$

SINCE THE POINT P (7, –3) IS ON THE CIRCLE, YOU HAVE

$$(7-2)^2 + (-3-3)^2 = r^2.$$

THIS IMPLIES, $5^2 + (-6)^2 = r^2$.

SO, $r^2 = 61$.

THEREFORE, THE EQUATION OF THE CIRCLE IS

$$(x-2)^2 + (y-3)^2 = 61.$$

Example 4 GIVE THE CENTRE AND RADIUS OF THE CIRCLE,

A $(x-5)^2 + (y+7)^2 = 64$ **B** $x^2 + y^2 + 6x - 8y = 0$.

Solution

A THE EQUATION IS $(x-5)^2 + (y+7)^2 = 8^2$. THEREFORE, THE CENTRE C OF THE CIRCLE IS C (5, –7) AND THE RADIUS r OF THE CIRCLE IS $r = 8$.

B BY COMPLETING THE SQUARE METHOD, THE EQUATION IS EQUIVALENT TO

$$x^2 + 6x + 9 + y^2 - 8y + 16 = 9 + 16 = 25.$$

THIS IS EQUIVALENT TO,

$$(x+3)^2 + (y-4)^2 = 5^2.$$

THEREFORE, THE CENTRE C OF THE CIRCLE IS C(–3, 4) AND THE RADIUS r OF THE CIRCLE IS $r = 5$.

ACTIVITY 3.5

1 FIND THE PERPENDICULAR DISTANCE FROM THE CENTRE OF THE CIRCLE WITH EQUATION

$$(x-1)^2 + (y+4)^2 = 16$$

TO EACH OF THE FOLLOWING LINES WITH EQUATIONS:

A $3x - 4y - 1 = 0$ **C** $3x - 4y + 2 = 0$

B $3x - 4y + 1 = 0$

2 SKETCH THE GRAPH OF THE CIRCLE AND EACH OF THE LINES GIVEN IN **QUESTION 1** ABOVE, IN THE SAME COORDINATE SYSTEM. WHAT DO YOU NOTICE?

FROM ACTIVITY 3.5, YOU MAY HAVE OBSERVED THAT:

1 IF THE PERPENDICULAR DISTANCE FROM THE CENTRE OF A CIRCLE TO A LINE IS LESS THAN THE RADIUS OF THE CIRCLE, THEN THE LINE INTERSECTS THE CIRCLE AT TWO POINTS. SUC CALLED A **secant** LINE TO THE CIRCLE.

2 IF THE PERPENDICULAR DISTANCE FROM THE CENTRE OF A CIRCLE TO A LINE IS EQUAL TO THE RADIUS OF THE CIRCLE, THEN THE LINE INTERSECTS THE CIRCLE AT ONLY ONE POINT. S IS CALLED A **tangent** LINE TO THE CIRCLE AND THE POINT OF INTERSECTION IS CALLED THE **point of tangency**.

3 IF THE PERPENDICULAR DISTANCE FROM THE CENTRE OF A CIRCLE TO A LINE IS GREATER THAN THE RADIUS OF THE CIRCLE, THEN THE LINE DOES NOT INTERSECT THE CIRCLE.

Note:

1 A LINE WITH EQUATION $Ax + By + C = 0$ INTERSECTS A CIRCLE WITH EQUATION $(x-h)^2+(y-k)^2=r^2$, IF AND ONLY IF,

$$\frac{|Ah+Bk+C|}{\sqrt{A^2+B^2}} \le r.$$

2 IF A LINE WITH EQUATION $Ax + By + C = 0$ INTERSECTS A CIRCLE WITH EQUATION $(x-h)^2+(y-k)^2=r^2$, THEN $(x-h)^2 + \left(-\frac{A}{B}x-\frac{C}{B}-k\right)^2 = r^2$ IS A QUADRATIC EQUATION IN x. IF $B = 0$, THEN $x = -\frac{C}{A}$ IS A VERTICAL LINE.

$(y-k)^2 = r^2 - \left(-\frac{C}{A}-h\right)^2 = r^2 - \left(\frac{C+hA}{A}\right)^2$, WHICH IS A QUADRATIC IN y.

SOLVING THIS EQUATION, YOU CAN GET POINT(S) OF INTERSECTION OF THE LINE AND THE

Example 5 FIND THE INTERSECTION OF THE CIRCLE WITH EQUATION $(x-1)^2 + (y+1)^2 = 25$ WITH EACH OF THE FOLLOWING LINES.

A $4x - 3y - 7 = 0$ **B** $x = 4$

Solution

A $4x - 3y - 7 = 0 \Leftrightarrow y = \frac{4x-7}{3}$

SO $(x-1)^2 + \left(\frac{4x-7}{3}+1\right)^2 = 25$

$\Rightarrow \quad (x-1)^2 + \left(\frac{4x-4}{3}\right)^2 = 25$

$\Rightarrow \quad 9(x-1)^2 + (4x-4)^2 = 225$

$\Rightarrow \quad 9(x^2 - 2x + 1) + (16x^2 - 32x + 16) = 225$

$\Rightarrow \quad 9x^2 - 18x + 9 + 16x^2 - 32x + 16 = 225$

$\Rightarrow \quad 25x^2 - 50x - 200 = 0$

$\Rightarrow \quad x^2 - 2x - 8 = 0$

$\Rightarrow \quad (x+2)(x-4) = 0$

$\Rightarrow \quad x = -2$ OR $x = 4$

THIS GIVES $y = -5$ AND $y = 3$, RESPECTIVELY.

HENCE THE LINE AND THE CIRCLE INTERSECT AT THE POINTS P(–2, –5) AND Q(4, 3).

B FOR THE LINE $x = 4$,

$\Rightarrow \quad (4-1)^2 + (y+1)^2 = 25$

$\Rightarrow \quad 9 + (y+1)^2 = 25$

$\Rightarrow \quad (y+1)^2 = 25 - 9 = 16$

$\Rightarrow \quad y + 1 = \pm 4$

$\Rightarrow \quad y = 3$ OR $y = -5$.

HENCE, THE INTERSECTION POINTS OF THE LINE AND THE CIRCLE ARE (4, 3) AND (4, –5).

Example 6 FOR THE CIRCLE $(x+1)^2 + (y-1)^2 = 13$, SHOW THAT $y = \frac{3}{2}x - 4$ IS A TANGENT LINE.

Solution THE DISTANCE FROM (–1, 1) TO THE LINE $-3x + 2y + 8 = 0$ IS

$$d = \frac{|-3(-1) + 2(1) + 8|}{\sqrt{(-3)^2 + 2^2}} = \frac{|13|}{\sqrt{13}} = \sqrt{13} = r$$

HENCE, $y = \frac{3}{2}x - 4$ IS A TANGENT LINE TO THE CIRCLE

$(x+1)^2 + (y-1)^2 = 13.$

Example 7 GIVE THE EQUATION OF THE LINE TANGENT TO THE CIRCLE WITH EQUATION $(x+1)^2 + (y-1)^2 = 13$ AT THE POINT P(–3, 4).

Solution First find the equation of the line ℓ that passes through the centre of the circle and the point of tangency.

The point of tangency is $T(x_0, y_0) = T(-3, 4)$ and the centre is $P(h, k) = P(-1, 1)$.

Therefore, equation of ℓ is given by:

$$\frac{y-y_0}{x-x_0} = \frac{y-4}{x-(-3)} = \frac{4-1}{-3+1}.$$

This implies, $\frac{y-4}{x+3} = -\frac{3}{2}$, which is equivalent to: $y - 4 = -\frac{3}{2}x - \frac{9}{2}$.

Hence $y = -\frac{3}{2}x - \frac{1}{2}$ is the equation of the line ℓ.

But the line ℓ is perpendicular to the tangent line to the circle at $T(-3, 4)$.

Therefore, the equation of the tangent line is given by:

$$\frac{y-4}{x-(-3)} = \frac{2}{3} \Rightarrow \frac{y-4}{x+3} = \frac{2}{3}$$

Therefore $y = \frac{2}{3}x + 6$ is equation of the tangent line to the circle at $(-3, 4)$.

✍Note:

✓ If a line ℓ is tangent to a circle $(x-h)^2 + (y-k)^2 = r^2$ at a point $T(x_o, y_o)$, then the equation of ℓ is given by

$$\frac{y-y_o}{x-x_o} = -\frac{x_o-h}{y_o-k}$$

Therefore, the equation of the tangent line in Example 7 can be found by:

$$\frac{y-y_o}{x-x_o} = \frac{y-4}{x+3} = -\left(\frac{-3+1}{4-1}\right) = \frac{2}{3}.$$

Example 8 Find the equation of the circle with centre $(2, 5)$ and the line with equation $x - y = 1$ is a tangent line to the circle.

Solution The distance from the centre $O(2, 5)$ of the circle to the line with equation $x - y - 1 = 0$ is the radius.

Thus, $r = \frac{|2-5-1|}{\sqrt{1^2+(-1)^2}} = 2\sqrt{2}$

Hence, the equation of the circle is $(x-2)^2 + (y-5)^2 = \left(2\sqrt{2}\right)^2 = 8$

Exercise 3.3

1 WRITE THE STANDARD FORM OF THE EQUATION OF A CIRCLE WITH THE GIVEN CENTRE AN

A C (–2, 3), $r = 5$ B C (8, 2), $r = \sqrt{2}$ C C (–2, –1), $r = 4$

2 FIND THE COORDINATES OF THE CENTRE AND THE RADIUS FOR EACH OF THE CIRCLES W EQUATIONS ARE GIVEN.

A $(x-2)^2 + (y-3)^2 = 7$ B $(x+7)^2 + (y+12)^2 = 36$

C $4(x+3)^2 + 4(y+2)^2 = 7$ D $(x-1)^2 + (y+3)^2 = 20$

E $x^2 + y^2 - 8x + 12y - 12 = 0$ F $x^2 + y^2 - 2x + 4y + 8 = 0$

3 WRITE THE EQUATION OF THE CIRCLE DESCRIBED BELOW:

A IT PASSES THROUGH THE ORIGIN AND HAS CENTRE AT (5, 2).

B IT IS TANGENT TO THE y-AXIS AND HAS CENTRE AT (3, –4).

C THE END POINTS OF ITS DIAMETER ARE (–2, –3) AND (4, 5).

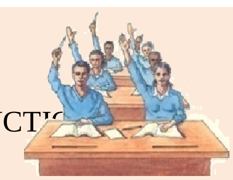
4 A CIRCLE HAS CENTRE AT (5, 12) AND IS TANGENT TO THE LINE WITH EQUATION $2x - y + 3 = 0$. WRITE THE EQUATION OF THE CIRCLE.

5 FIND THE EQUATION OF THE TANGENT LINE TO EACH CIRCLE AT THE INDICATED POINT.

A $x^2 + y^2 = 145$; P(9, –8) B $(x-2)^2 + (y-3)^2 = 10$; P(–1, 2)

3.2.3 Parabolas

ACTIVITY 3.6



1 DRAW THE GRAPH OF EACH OF THE FOLLOWING FUNCTI

A $y = x^2 + 2x + 3$ B $y = -x^2 + 5x - 4$

2 FIND THE AXIS OF SYMMETRY OF THE GRAPHS IN QUESTION 1 ABOVE.

FROM ACTIVITY 3.6, YOU HAVE SEEN THAT THE GRAPHS OF BOTH FUNCTIONS ARE PARABOLAS OPENS UPWARD AND THE OTHER OPENS DOWNWARD.

Definition 3.6

A **parabola** IS THE LOCUS OF POINTS ON A PLANE THAT HAVE THE SAME DISTANCE FROM GIVEN POINT AND A GIVEN LINE. THE POINT IS CALLED THE **focus** AND THE LINE IS CALLED THE **directrix** OF THE PARABOLA.

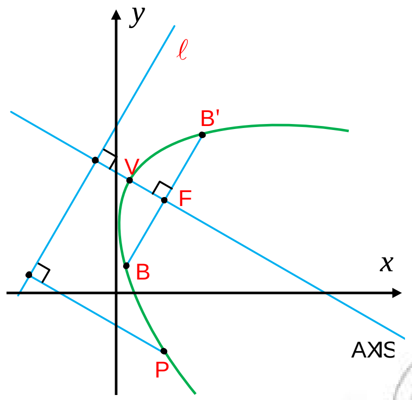


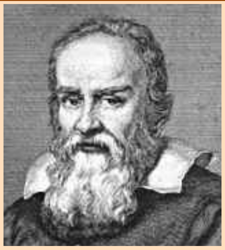
Figure 3.9

CONSIDER FIGURE 3.9. HERE ARE SOME TERMINOLOGIES FOR PARABOLAS.

- ✓ F IS THE **focus** OF THE PARABOLA.
- ✓ THE LINE ℓ IS THE **directrix** OF THE PARABOLA.
- ✓ THE LINE WHICH PASSES THROUGH THE FOCUS F AND IS PERPENDICULAR TO THE DIRE IS CALLED THE **axis** OF THE PARABOLA.
- ✓ THE POINT V ON THE PARABOLA WHICH LIES ON THE AXIS OF THE PARABOLA IS CALLEI **vertex** OF THE PARABOLA.
- ✓ THE CHORD $\overline{BB'}$ THROUGH THE FOCUS AND PERPENDICULAR TO THE AXIS IS CALLED THE **latus rectum** OF THE PARABOLA.
- ✓ THE DISTANCE $p = VF$ FROM THE VERTEX TO THE FOCUS IS CALLED THE **focal length** OF THE PARABOLA.

Historical Note

Galileo Galili (1564-1642)



In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. More recently, parabolic shapes have been used in designing automobile highlights, reflecting telescopes and suspension bridges.

NOW YOU ARE GOING TO SEE HOW TO FIND EQUATION OF A PARABOLA WITH ITS AXIS OF SY PARALLEL TO ONE OF THE COORDINATE AXES. THERE ARE TWO CASES TO CONSIDER. THE FI WHEN THE AXIS OF THE PARABOLA IS PARALLEL TO THE x-AXIS AND THE SECOND CASE IS WHEN THE AXIS OF THE PARABOLA IS PARALLEL TO THE y-AXIS.

Equation of a parabola whose axis is parallel to the *x*-axis

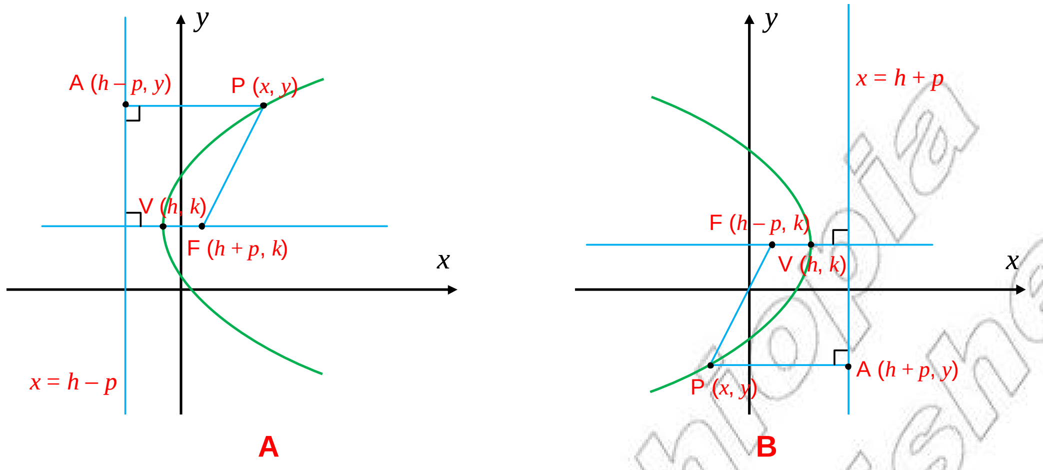


Figure 3.10

LET $V(h,k)$ BE THE VERTEX OF THE PARABOLA. THE AXIS OF THE PARABOLA IS THE LINE $y = k$.

IF THE FOCUS OF THE PARABOLA IS TO THE RIGHT OF THE VERTEX OF THE PARABOLA, THEN $F(h+p,k)$ AND THE EQUATION OF THE DIRECTRIX IS $x = h - p$. LET $P(x, y)$ BE A POINT ON THE PARABOLA. THEN THE DISTANCE FROM P TO F IS EQUAL TO THE DISTANCE FROM P TO THE I THAT IS, $PF = PA$ WHERE $A(h-p, y)$.

THIS IMPLIES, $\sqrt{(x-(h+p))^2+(y-k)^2} = \sqrt{(x-(h-p))^2+(y-y)^2}$.

SQUARING BOTH SIDES GIVES YOU $(x-(h+p))^2 + (y-k)^2 = (x-(h-p))^2$.

THIS IMPLIES, $x^2 - 2x(h+p)+(h+p)^2+(y-k)^2 = x^2 - 2x\,(h-p) + (h-p)^2$.

THIS CAN BE SIMPLIFIED TO THE FORM

$$(y-k)^2 = 4p(x-h)$$

THIS EQUATION IS CALLED THE **standard form of equation of a parabola** WITH VERTEX $V(h,\ k)$, FOCAL LENGTH p, THE FOCUS F IS TO THE RIGHT OF THE VERTEX AND ITS AXIS IS PARALLEL THE x-AXIS. THE PARABOLA OPENS TO THE RIGHT.

IF THE FOCUS OF THE PARABOLA IS TO THE LEFT OF THE VERTEX OF THE PARABOLA, THEN TH $F(h-p,k)$ AND THE EQUATION OF THE DIRECTRIX $x = h + p$. WITH THE SAME PROCEDURE AS ABOVE, YOU CAN GET THE EQUATION

$$(y-k)^2 = -4p\,(x-h)$$

THIS EQUATION IS CALLED THE **standard form of the equation of a parabola** WITH VERTEX $V(h, k)$, FOCAL LENGTH p; THE FOCUS F IS TO THE LEFT OF THE VERTEX AND ITS AXIS IS PARALLEL TO THE x-AXIS. IN THIS CASE, THE GRAPH OF THE PARABOLA OPENS TO THE LEFT.

THE STANDARD FORM OF THE EQUATION OF A PARABOLA WITH VERTEX (h, k) AND WHOSE AXIS IS PARALLEL TO x-AXIS IS GIVEN BELOW. SUCH A PARABOLA IS CALLED AN x-PARABOLA.

✍Note:

THE EQUATION

$$(y-k)^2 = \pm 4p\,(x-h)$$

REPRESENTS A PARABOLA WITH:

- ✓ VERTEX $V(h, k)$
- ✓ FOCUS $(h \pm p, k)$.
- ✓ DIRECTRIX: $x = h \pm p$.
- ✓ AXIS OF SYMMETRY: $y = k$
- ✓ IF THE SIGN IN FRONT OF p IS POSITIVE, THEN THE PARABOLA OPENS TO THE RIGHT.
- ✓ IF THE SIGN IN FRONT OF p IS NEGATIVE, THEN THE PARABOLA OPENS TO THE LEFT.

Example 9 FIND THE EQUATION OF THE DIRECTRIX, THE FOCUS OF THE PARABOLA, THE LENGTH OF THE LATUS RECTUM AND DRAW THE GRAPH OF THE PARABOLA.

$$y^2 = 4x$$

Solution THE VERTEX IS AT (0,0) AND $4p = 4$. HENCE $p = 1$.

THE PARABOLA OPENS TO THE RIGHT WITH FOCUS $(h + p, k) = (0 + 1, 0) = (1, 0)$ AND THE DIRECTRIX $x = h - p = 0 - 1 = -1$. THE AXIS OF THE PARABOLA IS THE x-AXIS.

THE LATUS RECTUM PASSES THROUGH THE FOCUS F(1, 0) AND IS PERPENDICULAR TO THE x-AXIS, THAT IS THE y-AXIS.

THEREFORE, THE EQUATION OF THE LINE CONTAINING THE LATUS RECTUM IS $x = 1$.

TO FIND THE ENDPOINTS OF THE LATUS RECTUM, YOU HAVE TO FIND THE INTERSECTION OF THE LINE $x = 1$ AND THE PARABOLA. THAT IS, $y^2 = 4\times 1 = 4 \Leftrightarrow y = \pm 2$.

THEREFORE, THE END POINTS OF THE LATUS RECTUM ARE $(1, -2)$ AND $(1, 2)$ AND THE LENGTH OF THE LATUS RECTUM IS:

$$\sqrt{(1-1)^2 + (-2-2)^2} = \sqrt{16} = 4.$$

THE GRAPH OF THE PARABOLA IS GIVEN IN FIGURE 3.11.

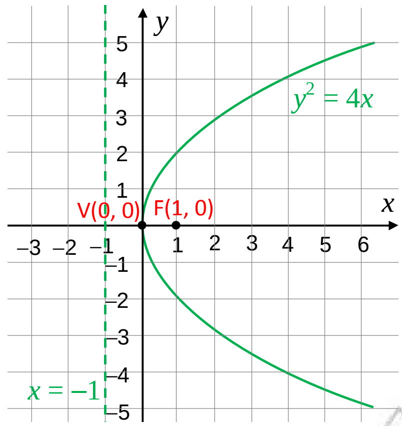


Figure 3.11

Example 10 FIND THE EQUATION OF THE DIRECTRIX AND THE FOCUS OF EACH PARABOLA AND DRAW THE GRAPH OF EACH OF THE FOLLOWING PARABOLAS.

A $4y^2 = -12x$ **B** $(y-2)^2 = 6(x-1)$ **C** $y^2 - 6y + 8x + 25 = 0$

Solution

A THE EQUATION $4y^2 = -12x$ CAN BE WRITTEN AS $y^2 = \frac{-12x}{4} = -3x$

THE VERTEX IS $V(h, k) = V(0, 0)$. $-4p = -3$ AND $p = \frac{3}{4}$.

SINCE THE SIGN IN FRONT OF p IS NEGATIVE, THE PARABOLA OPENS TO THE LEFT.

THE DIRECTRIX IS $x = h + p = 0 + \frac{3}{4} = \frac{3}{4}$.

THE FOCUS IS $F(h-p, k) = F\left(0 - \frac{3}{4}, 0\right) = F\left(-\frac{3}{4}, 0\right)$.

THE GRAPH OF THE PARABOLA IS GIVEN IN FIGURE 3.12

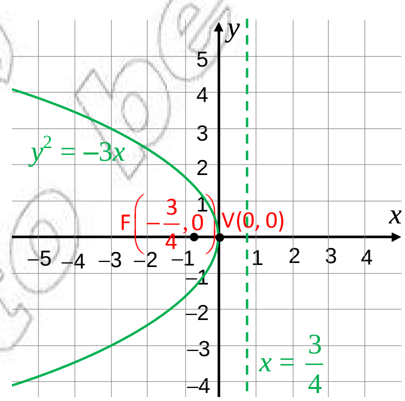


Figure 3.12

B THE VERTEX IS AT $V(h, k) = V(1, 2)$

SINCE $4p = 6$, THEN $p = \frac{6}{4} = \frac{3}{2}$. THE SIGN IN FRONT OF p IS POSITIVE. HENCE THE PARABOLA OPENS TO THE RIGHT.

THE FOCUS IS $F(h + p, k) = F\left(1+\frac{3}{2}, 2\right) = F\left(\frac{5}{2}, 2\right)$

THE DIRECTRIX IS $x = h - p = 1-\frac{3}{2} = \frac{-1}{2}$. THE AXIS OF THE PARABOLA IS THE HORIZONTAL LINE $y = k$, I.E. $y = 2$ AND THE GRAPH OF THE PARABOLA IS GIVEN IN FIGURE 3.13.

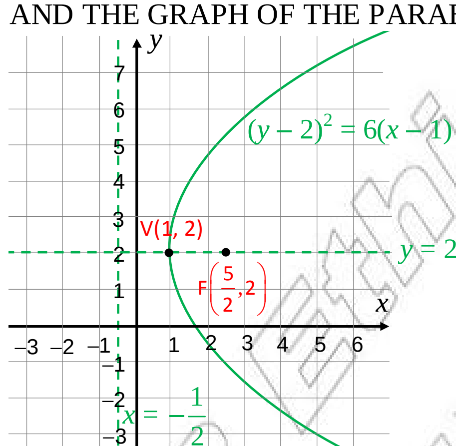


Figure 3.13

C BY COMPLETING THE SQUARE, THE EQUATION $y^2 - 6y + 8x + 25 = 0$ IS EQUIVALENT TO THE EQUATION $(y - 3)^2 = -8 (x + 2)$. THE VERTEX OF THE PARABOLA IS AT $V(h, k) = V(-2, 3)$ AND $-4p = -8$ IMPLIES $p = 2$. THE SIGN IN FRONT OF p IS NEGATIVE. HENCE THE PARABOLA OPENS TO THE LEFT.

THE FOCUS $F(h - p, k) = F(-2 -2,3) = F(-4, 3)$, THE EQUATION OF THE DIRECTRIX IS $x = h + p = -2 + 2 = 0$ AND THE EQUATION OF THE AXIS OF THE PARABOLA IS $y = k$, I.E. $y = 3$ WITH ITS GRAPH GIVEN IN FIGURE 3.14.

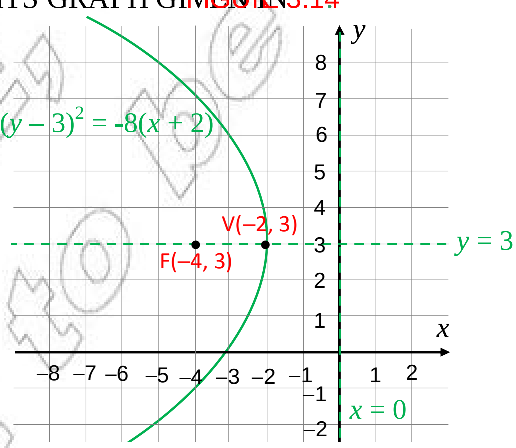


Figure 3.14

Example 11 FIND THE EQUATION OF THE PARABOLA WITH VERTEX V(–1, 4) AND FOCUS F(5, 4).

Solution HERE V(h, k) = V(–1, 4).

HENCE h = –1 AND k = 4 AND THE FOCUS IS GIVEN BY F(h + p, k) = F(5, 4).

THIS IMPLIES h + p = 5 AND k = 4. THEN, –1 + p = 5, WHICH IMPLIES p = 6

SINCE THE FOCUS F IS TO THE RIGHT OF THE VERTEX V, THE PARABOLA OPENS TO THE RI
HENCE THE EQUATION OF THE PARABOLA IS GIVEN BY:

$$(y-4)^2 = 24(x+1)$$

Equation of a parabola whose axis is parallel to the *y*-axis

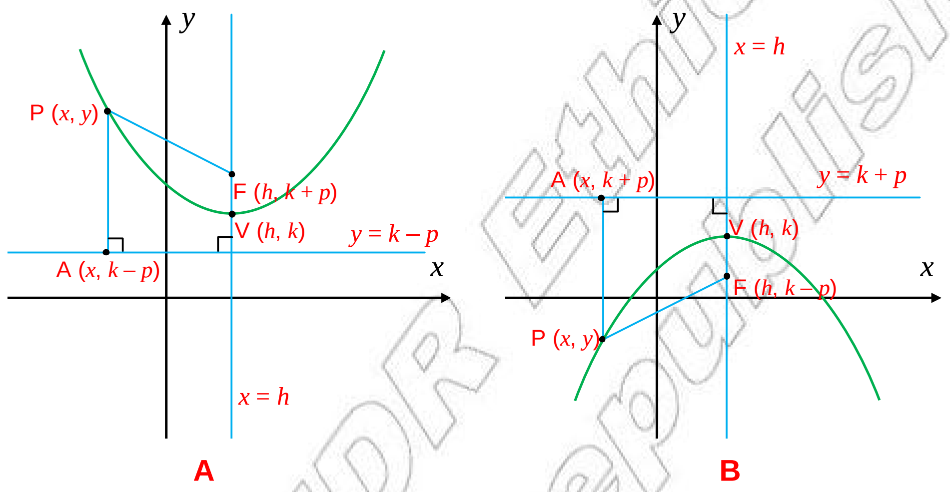


Figure 3.15

LET $V(h,k)$ BE THE VERTEX OF THE PARABOLA. THE AXIS OF THE PARABOLA IS THE LINE $x = h$.

IF THE FOCUS OF THE PARABOLA IS ABOVE THE VERTEX OF THE PARABOLA, THEN THE
$F(h, k+p)$ AND THE EQUATION OF THE DIRECTRIX IS $y = k - p$. LET $P(x, y)$ BE A POINT ON THE PARABOLA. THEN THE DISTANCE FROM P TO F IS EQUAL TO THE DISTANCE FROM P TO THE I
THAT IS, $PF = PA$ WHERE $A(x, k-p)$, AS SHOWN IN FIGURE 3.15

THIS IMPLIES $\sqrt{(x-h)^2 + (y-(k+p))^2} = \sqrt{(x-x)^2 + (y-(k-p))^2}$.

THIS CAN BE SIMPLIFIED TO THE FORM

$$(x-h)^2 = 4p\,(y-k)$$

THE STANDARD FORM OF EQUATION OF A PARABOLA WITH VERTEX $V(h, k)$ AND WHOSE AXIS IS PARALLEL TO THE y-AXIS. SUCH A PARABOLA IS CALLED A y-PARABOLA.

✍ Note:

THE EQUATION

$$(x-h)^2 = \pm 4p\,(y-k)$$

REPRESENTS A PARABOLA WITH

- ✓ VERTEX $V(h, k)$
- ✓ FOCUS $F(h, k \pm p)$.
- ✓ DIRECTRIX $y = k \mp p$.
- ✓ AXIS OF SYMMETRY $x = h$
- ✓ IF THE SIGN IN FRONT OF p IS POSITIVE, THEN THE PARABOLA OPENS UPWARD.
- ✓ IF THE SIGN IN FRONT OF p IS NEGATIVE, THEN THE PARABOLA OPENS DOWNWARD.

Example 12 FIND THE VERTEX, FOCUS AND DIRECTRIX OF THE FOLLOWING PARABOLAS; SKETCH THE GRAPHS OF THE PARABOLAS IN A AND B.

A $x^2 = 16y$ **B** $-2x^2 = 8y$

C $(x-2)^2 = 8(y+1)$ **D** $x^2 + 12y - 2x - 11 = 0$

Solution

A HERE $4p = 16$ IMPLIES $p = 4$.

SINCE THE SIGN IN FRONT OF p IS POSITIVE, THE PARABOLA OPENS UPWARD.

THE VERTEX IS $V(h, k) = V(0, 0)$.

THE FOCUS IS $F(0, p) = F(0, 4)$.

THE DIRECTRIX IS $y = k - p = 0 - 4 = -4$.

B $-2x^2 = 8y$ CAN BE WRITTEN AS $x^2 = -4y$.

HERE, $-4p = -4$ IMPLIES $p = 1$.

SINCE THE SIGN IN FRONT OF p IS NEGATIVE, THE PARABOLA OPENS DOWNWARD AS SHOWN IN FIGURE 3.16.

THE VERTEX IS $V(h, k) = V(0, 0)$.

THE FOCUS IS $F(h, k - p) = F(0, 0 - 1) = F(0, -1)$.

THE DIRECTRIX IS $y = k + p = 0 + 1 = 1$.

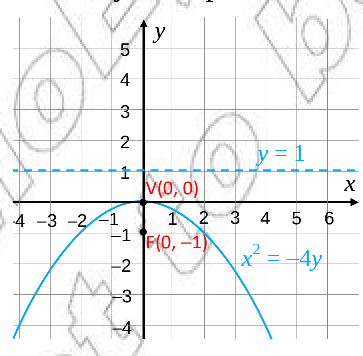


Figure 3.16

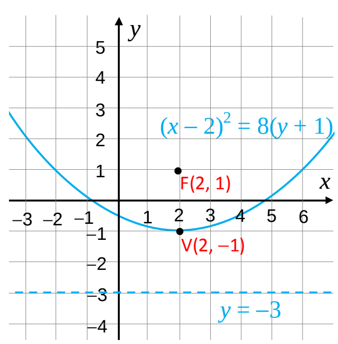


Figure 3.17

C HERE $4p = 8$ IMPLIES $p = 2$.

SINCE THE SIGN IN FRONT OF p IS POSITIVE, THE PARABOLA OPENS UPWARD AS SHOWN IN FIGURE 3.17.

THE VERTEX

$$V(h, k) = V(2, -1).$$

THE FOCUS IS

$$F(h, k + p) = F(2, -1 + 2) = F(2, 1).$$

THE DIRECTRIX IS $y = k - p = -1 - 2 = -3$.

D THE EQUATION $x^2 + 12y - 2x - 11 = 0$ IS EQUIVALENT TO $(x-1)^2 = -12(y-1)$.

HENCE $-4p = -12$ IMPLIES $p = 3$;

SINCE THE SIGN IN FRONT OF P IS NEGATIVE, THE PARABOLA OPENS DOWNWARD.

THE VERTEX IS $V(h, k) = V(1, 1)$

THE FOCUS IS $F(h, k - p) = F(1, 1 - 3) = F(1, -2)$

THE DIRECTRIX IS $y = k + p = 1 + 3 = 4$

Example 13 (*Parabolic reflector*)

A PARABOLOID IS FORMED BY REVOLVING A PARABOLA ABOUT ITS AXIS. A SPOTLIGHT IN THE FORM OF A PARABOLOID 6 INCHES DEEP HAS ITS FOCUS 3 INCHES FROM THE VERTEX. FIND THE RADIUS r OF THE OPENING OF THE SPOTLIGHT.

Solution FIRST LOCATE A PARABOLIC CROSS SECTION CONTAINING THE AXIS IN A COORDINATE SYSTEM AND LABEL THE KNOWN PARTS AND PARTS TO BE FOUND AS SHOWN IN FIGURE 3.18.

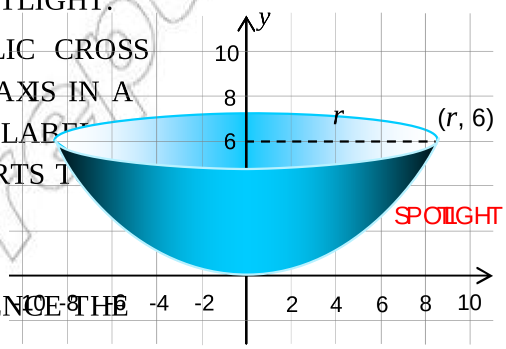


Figure 3.18

THE PARABOLA HAS THE y-AXIS AS ITS AXIS AND THE ORIGIN AS ITS VERTEX. HENCE THE EQUATION OF THE PARABOLA IS:

$$x^2 = 4py.$$

THE FOCUS IS GIVEN F(0, 3) = F(0, p). THUS $p = 3$ AND THE EQUATION OF THE PARABOLA IS:

$$x^2 = 12y.$$

THE POINT (r, 6) IS ON THE PARABOLA.

$$\Rightarrow r^2 = 12 \times 6$$

$$\Rightarrow r^2 = 72$$

$$\Rightarrow r = \sqrt{72} \cong 8.49 \text{ INCHES.}$$

Exercise 3.4

1 WRITE THE EQUATION OF EACH PARABOLA GIVEN BELOW.

A VERTEX(–2, 5); FOCUS (–2, –8) B VERTEX(–3, 4); FOCUS (–3, 12)

C VERTEX(4, 6); FOCUS (–8, 6) D VERTEX(–1, 8); FOCUS (6, 8)

2 NAME THE VERTEX, FOCUS AND DIRECTRIX OF THE PARABOLA WHOSE EQUATION IS GIV SKETCH THE GRAPH OF EACH OF THE FOLLOWING.

A $x^2 = 2y$ B $(x + 2)^2 = 4(y - 6)$

C $(y + 2)^2 = -16(x - 3)$ D $(x - 3)^2 = 4y$

3 WRITE THE EQUATION OF EACH PARABOLA DESCRIBED BELOW.

A FOCUS (3, 5); DIRECTRIX $y = 3$ B VERTEX(–2, 1); AXIS $y = 1$; $p = 1$

C VERTEX(4, 3); PASSES THROUGH (5, 2), VERTICAL AXIS

D FOCUS (5, 0); $p = 4$; VERTICAL AXIS

4 WRITE THE EQUATION OF EACH PARABOLA DESCRIBED BELOW.

A VERTEX AT THE ORIGIN, AXIS ALONG THE x-AXIS, PASSING THROUGH A (3, 6)

B VERTEX AT (4, 2), AXIS PARALLEL TO THE x-AXIS, PASSING THROUGH A (8, 7)

C VERTEX AT (5, –3), AXIS PARALLEL TO THE y-AXIS, PASSING THROUGH B (1, 2)

5 THE PARABOLA HAS A MULTIT SCIENTIFIC APPLICATIONS. A RE TELESCOPE IS DESIGNED BY USI PROPERTY OF A PARABOLA:

If the axis of a parabolic mirror is pointed toward a star, the rays from the star, upon striking the mirror, will be reflected to the focus.

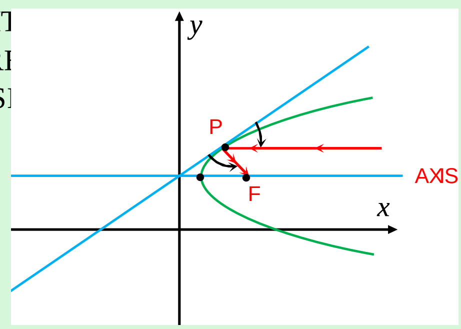


Figure 3.19

ANSWER THE FOLLOWING QUESTIONS

A A PARABOLIC REFLECTOR IS DESIGNED SO THAT ITS DIAMETER IS 12 M WHEN ITS I IS 4 M. LOCATE THE FOCUS.

B A PARABOLIC HEAD LIGHT LAMP IS DESIGNED IN SUCH A WAY THAT WHEN IT IS 16 WIDE IT HAS 6 CM DEPTH. HOW WIDE IS IT AT THE FOCUS?

6 FIND THE EQUATION OF THE PARABOLA DETERMINED BY THE GIVEN DATA.

A THE VERTEX IS AT (1,2), THE AXIS IS PARALLEL TO THE x-AXIS AND THE PARABOLA PASSES THROUGH (6,3).

B THE FOCUS IS AT (3,4), THE DIRECTRIX IS AT $x = 8$

3.2.4 Ellipses

Group Work 3.3

DO THE FOLLOWING IN GROUPS.

1 DRAW A CIRCLE OF RADIUS 5 CM.

2 USING TWO DRAWING PINS, A LENGTH OF A STRING AND A PENCIL DO THE FOLLOWIN THE PINS INTO A PAPER AT TWO POINTS. TIE THE STRING INTO A LOOSE LOOP AROUND PINS. PULL THE LOOP TAUT WITH THE PENCIL'S TIP SO AS TO FORM A TRIANGLE. MO PENCIL AROUND WHILE KEEPING THE STRING TAUT.

3 WHAT DO YOU OBSERVE FROM THE TWO DRAWINGS?

Definition 3.7

AN **ellipse** IS THE LOCUS OF ALL POINTS IN THE PLANE SUCH THAT THE SUM OF THE DISTA FROM TWO GIVEN FIXED POINTS IN THE PLANE, CALLED THE **foci**, IS CONSTANT.

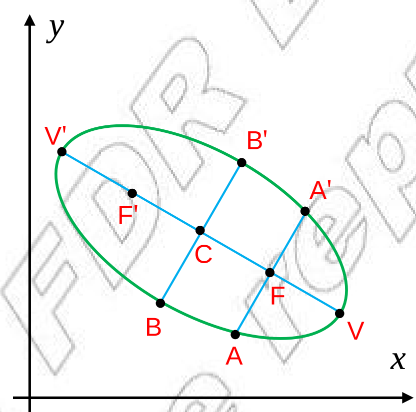


Figure 3.20

CONSIDER FIGURE 3.20. HERE ARE SOME TERMINOLOGIES FOR ELLIPSES.

✓ F and F' are **foci**.

✓ V, V', B AND B' ARE CALLED **vertices** OF THE ELLIPSE.

✓ $\overline{V'V}$ IS CALLED **the major axis** AND $\overline{B'B}$ IS CALLED THE **minor axis**.

✓ C, WHICH IS THE INTERSECTION POINT OF THE MAJOR AND MINOR AXES IS CALLED THE **centre** OF THE ELLIPSE.

- $\overline{CV}$ AND $\overline{CV'}$ ARE CALLED **semi-major axes** and $\overline{CB}$ AND $\overline{CB'}$ ARE CALLED **semi-minor axes.**
- Chord $\overline{AA'}$ WHICH IS PERPENDICULAR TO THE MAJOR AXIS AT F IS CALLED THE **latus rectum** OF THE ELLIPSE.
- THE DISTANCE FROM THE CENTRE TO A FOCUS IS DENOTED BY C.
- THE LENGTH OF THE **semi-major** AXIS IS DENOTED BY a AND THE LENGTH OF THE **semi-minor axis** IS DENOTED BY b.
- THE ECCENTRICITY OF AN ELLIPSE, USUALLY DENOTED BY e, IS THE RATIO OF THE DISTANCE BETWEEN THE TWO FOCI TO THE LENGTH OF THE MAJOR AXIS, THAT IS,

$$e = \frac{\text{DISTANCE BETWEEN THE TWO FOCI}}{\text{LENGH OF THE MAJOR AXIS}} = \frac{c}{a}$$

WHICH IS A NUMBER BETWEEN 0 AND 1.

NOTE THAT V'F' = VF AND VF + VF' = VV' = $2a$ ACCORDING TO THE DEFINITION. IF P IS ANY POINT ON THE ELLIPSE, YOU HAVE,

$$PF + PF' = 2a$$

SINCE B IS ON THE ELLIPSE, YOU ALSO HAVE THAT BF + BF' = $2a$. BUT BF = BF'. THIS IMPLIES BF = a. BY USING PYTHAGORAS THEOREM FOR RIGHT ANGLED TRIANGLE BCF, YOU GET,

$$CB^2 + CF^2 = BF^2$$

BUT CB = b, CF = c AND BF = a. THEREFORE, a, b AND c HAVE THE RELATION,

$$b^2 + c^2 = a^2$$

HISTORICAL NOTE

Johannes Kepler (1571-1630)

In the 17th century, Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one focus, (*his first law of planetary motion*).

Equation of an ellipse whose centre is at the origin

THERE ARE TWO CASES TO CONSIDER.

ONE OF THESE CASES IS WHERE THE MAJOR AXIS OF THE ELLIPSE IS PARALLEL TO THE x-AXIS AS SHOWN IN FIGURE 3.21 BELOW.

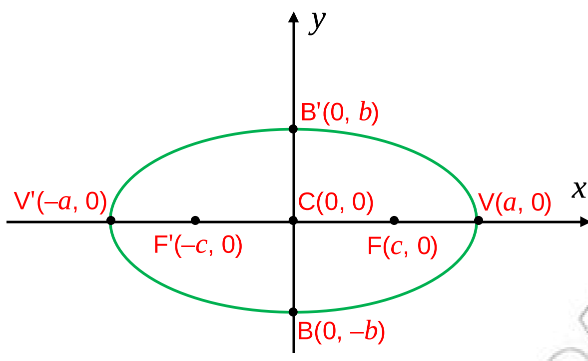


Figure 3.21

FROM THE DISCUSSION SO FAR, YOU HAVE,

$$PF' + PF = 2a.$$

THIS IMPLIES $\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$

$$\Rightarrow \sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

SQUARING BOTH SIDES GIVES YOU,

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2$$

THUS, $4a\sqrt{(x-c)^2+y^2} = 4a^2 + (x-c)^2 - (x+c)^2$

THIS IMPLIES $4a\sqrt{(x-C)^2+y^2} = 4a^2 + x^2 - 2xc + c^2 - x^2 - 2xc - c^2$

THIS GIVES YOU THE RESULT $a\sqrt{(x-c)^2+y^2} = a^2 - cx$

SQUARING BOTH SIDES GIVES

$$a^2((x-c)^2+y^2) = (a^2-cx)^2$$

$$\Rightarrow a^2(x^2 - 2xc + c^2 + y^2) = a^4 - 2a^2cx + c^2x^2$$

$$\Rightarrow a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2$$

$$\Rightarrow (a^2-c^2)x^2 + a^2y^2 = a^4 - a^2c^2$$

$$\Rightarrow (a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$$

FROM THE RELATION $a^2 = b^2 + c^2$, YOU GET, $a^2 - c^2 = b^2$.

THIS GIVES YOU,

$$b^2x^2 + a^2y^2 = a^2b^2$$

BY DIVIDING BOTH SIDES BY a^2b^2, YOU HAVE

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

THIS EQUATION IS CALLED THE **standard form of an equation of an ellipse** WHOSE MAJOR AXIS IS HORIZONTAL AND CENTRE IS AT (0, 0).

Example 14 GIVE THE COORDINATES OF THE FOCI OF THE ELLIPSE SHOWN BELOW. GIVE THE EQUATION OF THE ELLIPSE AND FIND THE ECCENTRICITY OF THE ELLIPSE.

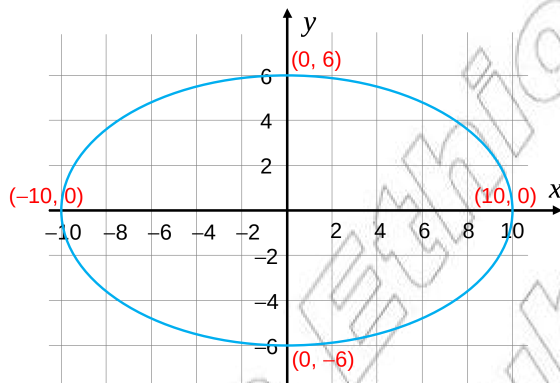


Figure 3.22

Solution FROM THE GRAPH OBSERVE THAT $a = 10$, AND $b = 6$. SINCE $a^2 = b^2 + c^2$, THEN $100 = 36 + c^2$. HENCE $c^2 = 64$. THIS IMPLIES $c = 8$.

THEREFORE, THE CENTRE IS C (0, 0) AND THE FOCI ARE F'($-c$, 0) = F'(–8, 0) AND F(c, 0) = F(8, 0) SINCE THE MAJOR AXIS IS HORIZONTAL.

THEN THE EQUATION OF THE ELLIPSE IS $\frac{x^2}{100}+\frac{y^2}{36}=1$.

THE ECCENTRICITY OF THE ELLIPSE IS $e=\frac{c}{a}=\frac{8}{10}=0.8$

Example 15 FIND THE EQUATION OF THE ELLIPSE WITH FOCI F'(–2, 0) AND F (2, 0), $a = 7$.

Solution F'(–2, 0) AND F (2, 0), IMPLIES THAT C (0, 0) AND $c = 2$. THE MAJOR AXIS OF ELLIPSE IS HORIZONTAL.

FROM THE RELATION $a^2 = b^2 + c^2$, YOU GET $b^2 = a^2 - c^2 = 7^2 - 2^2 = 45$.

HENCE, THE EQUATION OF THE ELLIPSE IS $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, OR $\frac{x^2}{49}+\frac{y^2}{45}=1$.

Equation of an ellipse whose centre is C (*h*, *k*) different from the origin

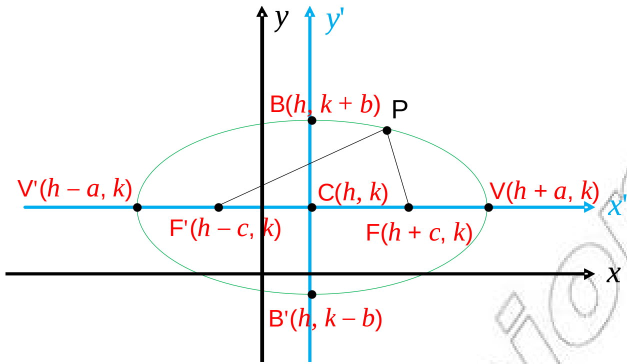


Figure 3.23

LET C(*h*, *k*) BE THE CENTRE OF THE ELLIPSE. CONSTRUCT A NEW *x'y'*-COORDINATE SYSTEM WITH ORIGIN AT C(*h*, *k*). THEN, FOR ANY POINT P ON THE ELLIPSE WITH COORDINATES (*x*, *y*) IN THE *xy*-COORDINATE SYSTEM AND (*x'*, *y'*) IN THE NEW *x'y'*-COORDINATE SYSTEM,

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

BUT THEN FROM TRANSLATION FORMULAE YOU HAVE $x' = x - h$ AND $y' = y - k$, WHICH GIVES

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

WHICH IS THE STANDARD EQUATION OF AN ELLIPSE WITH CENTRE C(*h*, *k*) AND MAJOR AXIS PARALLEL TO THE *x*-AXIS.

SIMILARLY, WHEN THE MAJOR AXIS IS VERTICAL, THE STANDARD EQUATION OF THE ELLIPSE IS

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \text{ WHEN C } (0, 0) \text{ AND } \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1, \text{ WHEN C}(h, k)$$

Example 16 FIND THE COORDINATES OF THE CENTRE, FOCI, THE LENGTH OF THE MAJOR AND MINOR AXES, DRAW THE GRAPH OF THE ELLIPSE, FIND THE ECCENTRICITY OF THE ELLIPSE AND THE LENGTH OF THE LATUS RECTUM.

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{1} = 1$$

Solution THE CENTRE OF THE ELLIPSE IS C (2, 1) AND THE MAJOR AXIS IS HORIZONTAL. ALSO $a^2 = 9$ AND $b^2 = 1$, WHICH IMPLIES $a = 3$ AND $b = 1$. THEN THE LENGTH OF THE MAJOR AXIS IS 6 AND THE LENGTH OF THE MINOR AXIS IS 2. HENCE THE VERTICES ARE V'(–1, 1), V (5, 1), B'(2, 0) AND B (2, 2).

FROM THE RELATION $c^2 = a^2 - b^2$, YOU GET $c = 2\sqrt{2}$ AND THE FOCI ARE $F(2 - 2\sqrt{2},\ 1)$ AND $F'(2 + 2\sqrt{2},\ 1)$.

THE ECCENTRICITY OF THE ELLIPSE IS $e = \frac{c}{a} = \frac{2\sqrt{2}}{3}$.

THE LINES CONTAINING THE LATUS RECTUMS ARE VERTICAL LINES. THESE LINES ARE $x = 2 + \sqrt{2}$ AND $x = 2 - \sqrt{2}$. THE INTERSECTION POINTS OF THE LINE $x = 2 + \sqrt{2}$ AND THE ELLIPSE ARE GIVEN BY:

$$\frac{(2+\sqrt{2}-2)^2}{9} + \frac{(y-1)^2}{1} = 1.$$

SOLVING THIS GIVES YOU: $y = \frac{3 \pm \sqrt{7}}{3}$.

HENCE, THE END POINTS OF ONE OF THE LATUS RECTUMS ARE:

$$\left(2+\sqrt{2}, \frac{3\pm\sqrt{7}}{3}\right).$$

THEREFORE, THE LENGTH OF THE LATUS RECTUM IS $\frac{2\sqrt{7}}{3}$

THE GRAPH OF THE ELLIPSE IS GIVEN IN FIGURE 3.24.

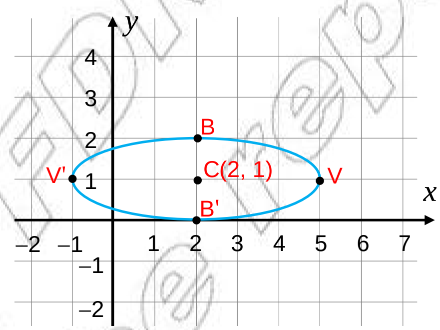


Figure 3.24

Example 17 FIND THE COORDINATES OF THE CENTRE, FOCI, THE LENGTH OF THE MAJOR AND MINOR AXES, DRAW THE GRAPH OF THE ELLIPSE .

$$\frac{(y+2)^2}{25} + \frac{(x+2)^2}{16} = 1$$

Solution THE CENTRE OF THE ELLIPSE IS C (–2, –2) AND THE MAJOR AXIS IS VERTICAL. ALSO $a^2 = 25$ AND $b^2 = 16$, WHICH IMPLIES $a = 5$ AND $b = 4$. SO THE LENGTH OF THE MAJOR AXIS IS 10 AND THE LENGTH OF THE MINOR AXIS IS 8 AND ALSO $c = \sqrt{a^2 - b^2} = 3$.

THEREFORE THE FOCI ARE $(h, k \pm c) = (-2, -2 \pm 3)$, THAT IS, $F'(-2, -5)$, $F(-2, 1)$

AND ALSO THE VERTICES ARE V'(–2, –7), V (–2, 3), B'(–6, –2), AND B (2, –2).

THE GRAPH OF THE ELLIPSE IS GIVEN IN FIGURE 3.25

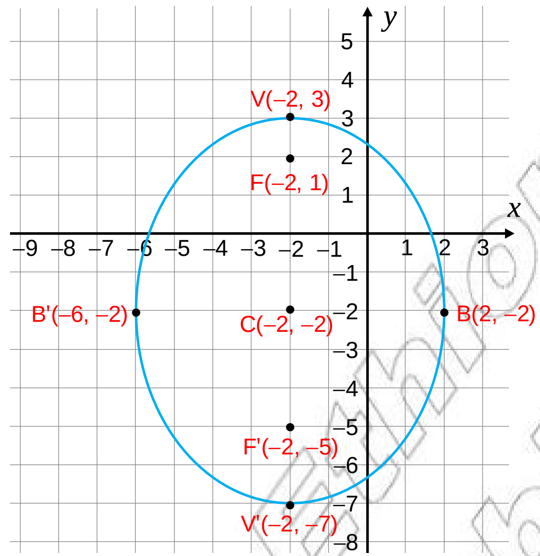


Figure 3.25

Exercise 3.5

1 WRITE THE EQUATION OF EACH ELLIPSE DESCRIBED BELOW.

A C (0, 0); $a = 6$, $b = 4$; HORIZONTAL MAJOR AXIS

B FOCI (–3, 0), (3, 0); $a = 8$

C C (0, 0); $a = 8$, $b = 6$; VERTICAL MAJOR AXIS

D C (5, 0); $a = 5$, $b = 2$; HORIZONTAL MAJOR AXIS

2 NAME THE CENTRE, THE FOCI AND THE VERTICES OF EACH ELLIPSE WHOSE EQUATION IS ALSO SKETCH THE GRAPH OF EACH ELLIPSE.

A $\dfrac{(x-3)^2}{25} + \dfrac{(y-4)^2}{16} = 1$ **B** $\dfrac{(y+2)^2}{25} + \dfrac{(x-1)^2}{4} = 1$

C $\dfrac{(y-2)^2}{25} + \dfrac{(x-3)^2}{5} = 1$

3 FIND THE EQUATION OF THE ELLIPSE WITH

A CENTRE AT (1, 4) AND VERTICES AT (10, 4) AND (1, 2)

B FOCI AT (–1, 0), (1, 0) AND THE LENGTH OF THE MAJOR AXIS 6 UNITS.

C VERTEX AT (6, 0), FOCUS AT (–1, 0) AND CENTRE AT (0, 0).

D CENTRE AT $\left(0, \frac{-1}{2}\right)$, FOCUS AT (0, 1) AND PASSING THROUGH (2, 2).

E CENTRE (0, 0), VERTEX (0, –5) AND LENGTH OF MINOR AXIS 8 UNITS.

4 THE PLANET MARS TRAVELS AROUND THE SUN IN AN ELLIPSE WHOSE EQUATION IS APPROXIMATELY GIVEN BY

$$\frac{x^2}{(228)^2} + \frac{y^2}{(227)^2} = 1$$

WHERE x AND y ARE MEASURED IN MILLIONS OF KILOMETRES. FIND

A THE DISTANCE FROM THE SUN TO THE OTHER FOCUS OF THE ELLIPSE (*in millions of kilometres*).

B HOW CLOSE MARS GETS TO THE SUN.

C THE GREATEST POSSIBLE DISTANCE BETWEEN MARS AND THE SUN.

3.2.5 Hyperbolas

Definition 3.8

A **hyperbola** IS DEFINED AS THE LOCUS OF POINTS IN THE PLANE SUCH THAT THE DIFFERE BETWEEN THE DISTANCES FROM TWO FIXED POINTS IS A CONSTANT. THE FIXED POINTS ARE **foci**. THE POINT MIDWAY BETWEEN THE FOCI IS CALLED THE **centre** OF THE HYPERBOLA.

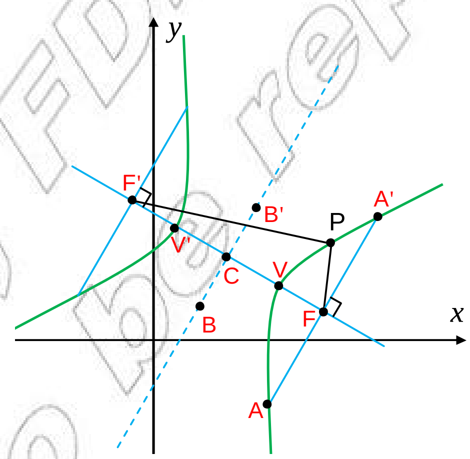


Figure 3.26

CONSIDER FIGURE 3.26. HERE ARE SOME TERMINOLOGIES FOR HYPERBOLAS.

- ✓ F AND F' ARE THE **foci** OF THE HYPERBOLA.
- ✓ C IS THE **centre** OF THE HYPERBOLA.

- THE POINTS V AND V' ON EACH BRANCH OF THE HYPERBOLA NEAREST TO THE CENTRE ARE CALLED **vertices**.
- $\overline{V'V}$ IS CALLED THE **transverse axis** OF THE HYPERBOLA AND CV = CV' IS DENOTED BY *a* AND CF = CF' IS DENOTED BY *c*.
- DENOTE $c^2 - a^2$ BY b^2, SO THAT $b = \sqrt{c^2 - a^2}$.
- THE SEGMENT OF SYMMETRY PERPENDICULAR TO THE TRANSVERSE AXIS AT THE CENTRE, WHICH HAS LENGTH 2*b*, IS CALLED THE **conjugate axis**.
- THE END POINTS B AND B' OF THE **conjugate axis** OF THE HYPERBOLA ARE CALLED CO-VERTICES.
- THE **eccentricity** OF THE HYPERBOLA, USUALLY DENOTED BY *e*, IS THE RATIO OF THE DISTANCE BETWEEN THE TWO FOCI TO THE LENGTH OF THE TRANSVERSE AXIS, THAT IS,

$$e = \frac{\text{DISTANCE BETWEEN THE TWO FOCI}}{\text{LENGH OF THE TRANSVERSE AXIS}} = \frac{c}{a}$$

  WHICH IS A NUMBER GREATER THAN 1.
- THE CHORDS WITH END POINTS ON THE HYPERBOLA PASSING THROUGH THE FOCI AND PERPENDICULAR TO $\overline{FF'}$ ARE CALLED THE **latus rectums**.

✍Note:

HYPERBOLAS OCCUR FREQUENTLY AS GRAPHS OF EQUATIONS IN CHEMISTRY, PHYSICS, BIO ECONOMICS (BOYLE'S LAW, OHM'S LAW, SUPPLY AND DEMAND CURVES).

Equation of a hyperbola with centre at the origin and whose transverse axis is horizontal

CONSIDER A HYPERBOLA WITH FOCI F(c, 0), F'($-c$, 0) AND CENTRE C (0, 0).

THEN, A POINT P(x, y) IS ON THE HYPERBOLA, IF AND ONLY IF

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a$$

ADDING $\sqrt{(x+c)^2+y^2}$ TO BOTH SIDES OF THE ABOVE EQUATION GIVES YOU

$$\sqrt{(x-c)^2+y^2} = \pm 2a + \sqrt{(x+c)^2+y^2} .$$

BY SQUARING BOTH SIDES YOU HAVE,

$$(x-c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2+y^2} + (x+c)^2 + y^2.$$

THIS IMPLIES $\pm 4a\sqrt{(x+c)^2+y^2} = 4a^2 + x^2 + 2xc + c^2 - x^2 + 2xc - c^2$

THAT IS, $\pm 4a\sqrt{(x+c)^2+y^2} = 4a^2 + 4xc$.

THIS IMPLIES, $\pm a\sqrt{(x+c)^2+y^2} = a^2 + xc$.

AGAIN SQUARING BOTH SIDES OF THE ABOVE EQUATION GIVES YOU:

$$a^2\ ((x+c)^2 + y^2) = a^4 + 2a^2xc + x^2c^2$$

THIS IMPLIES, $(a^2 - c^2)\, x^2 + a^2 y^2 = a^2\,(a^2 - c^2)$.

RECALL THAT $c^2 - a^2 = b^2$. THUS, $-b^2 x^2 + a^2 y^2 = -a^2 b^2$, WHICH REDUCES TO

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

THIS EQUATION IS CALLED THE **standard form of equation of a hyperbola** WITH CENTRE AT C(0, 0) AND TRANSVERSE AXIS HORIZONTAL.

Example 18 FIND THE EQUATION OF A HYPERBOLA, IF THE FOCI ARE F (2, 5) AND F'(−4, 5) AND THE TRANSVERSE AXIS IS 4 UNITS LONG. DRAW THE GRAPH OF THE HYPERBO

Solution THE MID-POINT OF $\overline{FF'}$ IS THE CENTRE OF THE HYPERBOLA AND IT IS C (−1, 5). THE TRANSVERSE AXIS IS $2a = 4$. SO, $a = 2$ AND FF' $= 2c = 6$.

BESIDES, SINCE F AND F' LIE ON A HORIZONTAL LINE, THE TRANSVERSE AXIS IS HORIZONT

USING THE RELATION $b^2 = c^2 - a^2 = 9 - 4 = 5$, THE EQUATION BECOMES

$$\frac{(x+1)^2}{4} - \frac{(y-5)^2}{5} = 1.$$

THE GRAPH OF THE HYPERBOLA IS GIVEN IN FIGURE 3.27

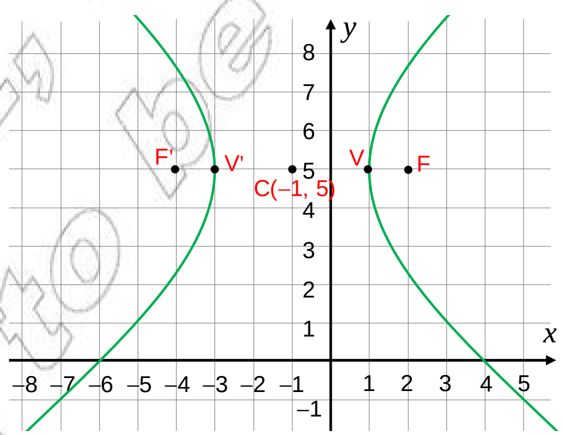
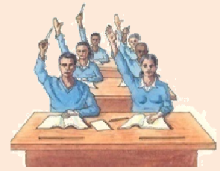


Figure 3.27

ACTIVITY 3.7



CONSIDER THE HYPERBOLA WITH EQUATION

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

AND ANSWER EACH OF THE FOLLOWING.

A DRAW THE GRAPH OF THE HYPERBOLA WITH THE EQUATION GIVEN ABOVE.

B MARK THE POINTS WITH COORDINATES $(\pm 3, 0)$ ON THE x-AXIS AND WITH COORDINATES $(0, \pm 4)$ ON THE y-AXIS.

C DRAW A RECTANGLE WITH SIDES PASSING THROUGH EACH OF THE POINTS IN **B** AND PARALLEL TO THE COORDINATE AXES.

D DRAW THE LINES THAT CONTAIN THE DIAGONALS OF THE RECTANGLE IN **C** ABOVE.

Asymptotes

IF A POINT P ON A CURVE MOVES FARTHER AND FARTHER AWAY FROM THE ORIGIN, AND THE BETWEEN P AND SOME FIXED LINE TENDS TO ZERO, THEN SUCH A LINE IS CALLED **an asymptote to the curve**.

FROM ACTIVITY 3.7 YOU MAY HAVE OBSERVED THAT THE LINES THROUGH THE DIAGONALS RECTANGLE THAT PASSES THROUGH POINTS WITH COORDINATES $(\pm 3, 0)$ ON THE x-AXIS AND $(0, \pm 4)$ ON THE y-AXIS AND PARALLEL TO THE COORDINATE AXES ARE ASYMPTOTES TO THE GRAPH HYPERBOLA WITH EQUATION

$$\frac{x^2}{9}-\frac{y^2}{16}=1.$$

CONSIDER THE HYPERBOLA WITH EQUATION

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

THIS EQUATION IS EQUIVALENT TO

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=1$$

OR

$$\frac{x}{a}-\frac{y}{b}=\frac{ab}{bx+ay}$$

ONE BRANCH OF THE HYPERBOLA LIES IN THE FIRST QUADRANT. IF A POINT P ON THE HYPER
MOVES FARTHER AND FARTHER AWAY FROM THE ORIGIN ON THIS BRANCH OF THE HYPERBO
x AND y BECOME INFINITE AND

$$\frac{ab}{bx+ay}$$

TENDS TO ZERO. THIS IMPLIES THE LINE

$$\frac{x}{a}-\frac{y}{b}=0 \quad \text{OR} \quad y=\frac{b}{a}x$$

IS AN ASYMPTOTE TO THE GRAPH OF THE HYPERBOLA.

BY SYMMETRY, THE LINE

$$\frac{x}{a}+\frac{y}{b}=0 \quad \text{OR } y=-\frac{b}{a}x$$

IS ALSO AN ASYMPTOTE TO THE GRAPH OF THE HYPERBOLA.

IF YOU INTERCHANGE x AND y IN THE EQUATION

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

THE NEW EQUATION BECOMES

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

AND REPRESENTS A HYPERBOLA WITH FOCI F(0, c) AND F'(0, $-c$), VERTICES V(0, a) AND V'(0, $-a$), CO-VERTICES B(b, 0) AND B'($-b$, 0), CENTRE C(0, 0), THE TRANSVERSE AXIS IS THE y-AXIS. IN THIS CASE, THE LINES $y=\pm\frac{a}{b}x$ ARE ASYMPTOTES TO THE GRAPH OF THE HYPERBOLA.

LET C(h, k) BE THE CENTRE OF THE HYPERBOLA. CONSTRUCT A NEW $x'y'$-COORDINATE SYSTEM WITH ORIGIN AT (h, k). THEN, FOR ANY POINT P ON THE HYPERBOLA WITH COORDINATES (x, y) IN THE xy-COORDINATE SYSTEM AND (x', y') IN THE NEW $x'y'$-COORDINATE SYSTEM,

$$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}=1$$

USING TRANSLATION FORMULA $x' = x - h$ AND $y' = y - k$, THIS REDUCES TO

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

WHICH IS THE STANDARD EQUATION OF A HYPERBOLA WITH CENTRE (h, k) AND TRANSVERSE AXIS PARALLEL TO THE x-AXIS.

SIMILARLY, WHEN THE TRANSVERSE AXIS IS VERTICAL, THE STANDARD EQUATION OF THE H
GIVEN BY:

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1, \text{ WHEN C } (0, 0) \text{ AND}$$

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1, \quad \text{WHEN C}(h, k)$$

THE FOLLOWING TABLE GIVES ALL POSSIBLE STANDARD FORMS OF EQUATIONS OF HYPERBO

Equation	Centre	Transverse axis	Asymptotes
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	(0, 0)	HORIZONTAL	$y=\pm\frac{b}{a}x$
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	(h, k)	HORIZONTAL	$y-k=\left(\pm\frac{b}{a}(x-h)\right)$
$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$	(0, 0)	VERTICAL	$y=\pm\frac{a}{b}x$
$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$	(h, k)	VERTICAL	$y-k=\left(\pm\frac{a}{b}(x-h)\right)$

Example 19 FIND ASYMPTOTES OF THE HYPERBOLA, IF THE FOCI ARE F(2, 5) AND F'(−4, 5) AND THE TRANSVERSE AXIS IS 4 UNITS LONG.

Solution FROM EXAMPLE 18, THE EQUATION OF THE HYPERBOLA IS:

$$\frac{(x+1)^2}{4}-\frac{(y-5)^2}{5}=1$$

THE ASYMPTOTES OF THE HYPERBOLA ARE:

$$y-k=\pm\left(\frac{b}{a}(x-h)\right).$$

THAT IS, $y-5=\pm\left(\frac{\sqrt{5}}{2}(x+1)\right) \Rightarrow y=\pm\left(\frac{\sqrt{5}}{2}(x+1)\right)+5$

WHICH GIVES THE LINES WITH EQUATIONS

$$y=\frac{\sqrt{5}}{2}x+\frac{\sqrt{5}+10}{2}, \text{ AND } y=-\frac{\sqrt{5}}{2}x+\frac{10-\sqrt{5}}{2}.$$

Example 20 FIND THE EQUATION OF THE HYPERBOLA WITH VERTICES (1, 4), AND (1, 2) AND $b = 2$.

Solution THE VERTICES LIE ON A VERTICAL LINE. THUS, THE EQUATION MUST BE OF THE FORM

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

THE CENTRE IS MID WAY BETWEEN (1, 2) AND (1, –2). SO, C(1, 0).

ALSO $2a$ = VV' = 4 $\Rightarrow a = 2$.

IT FOLLOWS THAT THE EQUATION IS $\frac{(y-0)^2}{4} - \frac{(x-1)^2}{4} = 1$

OR $\frac{y^2}{4} - \frac{(x-1)^2}{4} = 1$

Example 21 SKETCH THE HYPERBOLA WITH EQUATION:

$16y^2 - 9x^2 = 144$.

DRAW ITS ASYMPTOTES AND GIVE THE COORDINATES OF ITS VERTICES AND FOCI.

Solution THE EQUATION $16y^2 - 9x^2 = 144$ IS EQUIVALENT TO $\frac{16y^2}{144} - \frac{9x^2}{144} = 1$.

THEREFORE, THE EQUATION OF THE HYPERBOLA IS $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

THIS IMPLIES THE CENTRE IS C(0, 0), $a = 3$, $b = 4$, AND THE VERTICES OF THE HYPERBOLA ARE V'(0, –3), AND V(0, 3).

FROM THE RELATION $c^2 = a^2 + b^2 = 25$, YOU GET $c = 5$.

HENCE THE FOCI ARE F'$(h, k - c)$ AND F$(h, k + c)$, WHICH IMPLIES F' (0, –5) AND F (0, 5).

ASYMPTOTES OF THE HYPERBOLA ARE $y = \pm \frac{a}{b} x$, THAT IS, $y = \pm \frac{3}{4} x$.

THE GRAPH OF THE HYPERBOLA IS GIVEN IN FIGURE 3.28

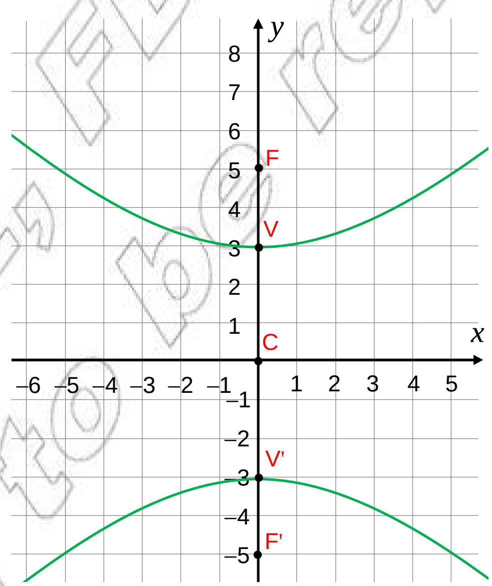


Figure 3.28

Exercise 3.6

1 FIND THE EQUATION OF EACH HYPERBOLA WITH THE INFORMATION GIVEN BELOW.

A CENTRE AT C(0, 0); $a = 8$, $b = 5$, HAVING HORIZONTAL TRANSVERSE AXIS.

B FOCI AT F(10, 0) AND F'(–10, 0); $2a = 16$.

C CENTRE C(–1, 4); $a = 2$, $b = 3$; VERTICAL TRANSVERSE AXIS.

D VERTICES V(2, 1), V'(–2, 1); $b = 2$.

2 NAME THE CENTRE, FOCI, VERTICES AND THE EQUATIONS OF THE ASYMPTOTES OF EACH HYPERBOLA GIVEN BELOW. ALSO SKETCH THEIR GRAPH.

A $\dfrac{x^2}{36} - \dfrac{y^2}{81} = 1$

B $\dfrac{(x+3)^2}{9} - \dfrac{(y+6)^2}{36} = 1$

C $\dfrac{y^2}{25} - \dfrac{x^2}{16} = 1$

D $\dfrac{(y-3)^2}{25} - \dfrac{(x-2)^2}{25} = 1$

3 WRITE THE EQUATION OF EACH HYPERBOLA SATISFYING THE FOLLOWING CONDITIONS:

A CENTRE C(4, –2); FOCUS F(7, –2); VERTEX V(6, –2)

B CENTRE C(4, 2); VERTEX V(4, 5); EQUATION OF ONE ASYMPTOTE IS $4y - 3x = -4$.

C VERTICES AT V(0, –4), V'(0, 4); FOCI AT F(0, –5), F'(0, 5)

D VERTICES AT V(–2, 3), V'(6, 3); ONE FOCUS AT F(–4, 3)

E THE TRANSVERSE AXIS COINCIDES WITH THE x-AXIS, CENTRE AT C(2, 0); LENGTHS OF TRANSVERSE AND CONJUGATE AXES EQUAL TO 8 AND 6, RESPECTIVELY.

F THE LENGTH OF THE TRANSVERSE AXIS IS EQUAL TO 8; THE END POINTS OF THE CONJUGATE AXIS ARE A(5, –5) AND B(5, 3).

4 A HYPERBOLA FOR WHICH $a = b$ IS CALLED **equilateral**. SHOW THAT A HYPERBOLA IS EQUILATERAL, IF AND ONLY IF ITS ASYMPTOTES ARE PERPENDICULAR TO EACH OTHER.

Key Terms

angle of inclination	major axis	slope-intercept form
asymptote	minor axis	tangent line
axis	parallel lines	translation formulas
centre	perpendicular lines	transverse axis
conjugate axis	point of tangency	two-point form
directrix	point-slope form	vertex
focal length	radius	x-intercept
focus	secant line	y-intercept
latus rectum	slope	

Summary

1. THE **slope** OF A LINE THROUGH (x_1, y_1) AND (x_2, y_2) IS GIVEN BY $\frac{y_2 - y_1}{x_2 - x_1}$.
2. **Two point form:** IF (x_1, y_1) AND (x_2, y_2) WITH $x_1 \neq x_2$ ARE GIVEN, THE LINE THROUGH THEM HAS AN EQUATION $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$
3. **Point-slope form:** IF A POINT (x_1, y_1) AND SLOPE m ARE GIVEN, THE EQUATION OF THE LINE IS $y - y_1 = m(x - x_1)$
4. **Slope-intercept form:** IF THE SLOPE m AND y-INTERCEPT b ARE GIVEN, THEN THE EQUATION OF THE LINE IS $y = mx + b$.
5. TWO LINES ARE **parallel** IF AND ONLY IF THEY HAVE THE SAME ANGLE OF INCLINATION.
6. THE SLOPE OF A **non-vertical** LINE IS TAN, WHERE IS THE ANGLE OF INCLINATION OF THE LINE, WITH 0 < 180°.
7. THE ANGLE BETWEEN TWO NON-VERTICAL LINES IS GIVEN BY THE FORMULA

   TAN = $\frac{m - n}{1 + mn}$, WHERE m AND n ARE THE SLOPES OF THE LINES.
8. TWO LINES ARE **perpendicular** IF AND ONLY IF THE ANGLE BETWEEN THEM IS 90°
9. IF TWO PERPENDICULAR LINES ARE NON-VERTICAL, THEN $mn = -1$, WHERE m AND n ARE THEIR SLOPES.
10. THE **general form of equation of a line** IS $Ax + By + C = 0$, WHERE $A \neq 0$ OR $B \neq 0$ ARE FIXED REAL NUMBERS.

11 THE DISTANCE FROM THE ORIGIN TO THE LINE $Ax + By + C = 0$ IS GIVEN BY $\dfrac{|C|}{\sqrt{A^2 + B^2}}$

12 THE DISTANCE FROM (h, k) TO $Ax + By + C = 0$ IS $\dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$

13 IF THE xy-COORDINATE SYSTEM IS TRANSLATED TO A NEW $x'y'$-COORDINATE SYSTEM WITH ORIGIN $O'(h, k)$, THEN THE TRANSLATION FORMULAE ARE

$$x' = x - h$$

$$y' = y - k$$

14 THE **standard form of the equation of a circle** IS $(x - h)^2 + (y - k)^2 = r^2$, WHERE (h, k) IS THE CENTRE AND r IS THE RADIUS.

15 THE LINE THAT TOUCHES A CIRCLE AT ONLY ONE POINT IS CALLED **tangent line** AND ITS EQUATION IS $\dfrac{y - y_0}{x - x_0} = \dfrac{-(x_0 - h)}{y_0 - k}$, WHERE (x_0, y_0) IS THE **point of tangency** AND (h, k) IS THE **centre** OF THE CIRCLE.

16 THE **standard equation of a parabola** IS EITHER

$(x - h)^2 = \pm 4p\,(y - k)$ (AXIS // TO THE y-AXIS)

OR $(y - k)^2 = \pm 4p\,(x - h)$ (AXIS // TO THE x-AXIS)

17 THE **standard equation of an ellipse** IS EITHER

$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ (MAJOR AXIS HORIZONTAL)

OR $\dfrac{(y-k)^2}{a^2} + \dfrac{(x-h)^2}{b^2} = 1$ (MAJOR AXIS VERTICAL)

WHERE $b^2 + c^2 = a^2$

18 THE **standard equation of a hyperbola** IS EITHER

$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ (TRANSVERSE AXIS HORIZONTAL)

OR $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ (TRANSVERSE AXIS VERTICAL)

WHERE $a^2 + b^2 = c^2$

19 THE EQUATIONS OF THE ASYMPTOTES OF A HYPERBOLA WITH HORIZONTAL TRANSVERSE AXIS ARE

$$y - k = \pm \frac{b}{a}(x - h)$$

AND THOSE WITH VERTICAL TRANSVERSE AXIS ARE $y - k = \pm \dfrac{a}{b}(x - h)$

Review Exercises on Unit 3

1 WRITE EACH OF THE FOLLOWING IN THE GENERAL FORM OF EQUATION OF A LINE.

A $y = -3$ **B** $x = 9$ **C** $y = \frac{1}{2}x + 4$

D $y - 3 = 4 - x$ **E** $3x = 7 - 4y$

2 GIVE THE EQUATION OF THE LINE THAT SATISFIES THE GIVEN CONDITIONS:

A PASSES THROUGH P(–2, 3) AND HAS SLOPE –1.

B PASSES THROUGH P(3, 7) AND Q(6, –10)

C PARALLEL TO THE LINE WITH EQUATION $y = 3x - 4$ AND PASSES THROUGH A(3, –2).

D PERPENDICULAR TO THE LINE WITH EQUATION $x = 2y - 4$ AND y-INTERCEPT 4.

3 FIND THE TANGENT OF THE ACUTE ANGLE BETWEEN THE FOLLOWING LINES:

A $2x + y - 2 = 0$
$3x + y + 1 = 0$

B $x - 6y + 5 = 0$
$2y - x - 1 = 0$

C $-x - 5y - 2 = 0$
$y - 4x + 7 = 0$

D $x - 6y + 5 = 0$
$2y - x - 1 = 0$

4 FIND THE DISTANCE FROM THE GIVEN POINT TO THE LINE WHOSE EQUATION IS GIVEN.

A P(4, 3); $2x - 3y + 2 = 0$ **B** A(0, 0); $2x - 3y + 2 = 0$

C Q(– 1, 0); $2x - 3y + 2 = 0$ **D** B(– 2, 4); $4y = 3x - 1$

5 FIND THE DISTANCE BETWEEN THE PAIRS OF PARALLEL LINES WHOSE EQUATIONS ARE BELOW:

A $2x - 3y + 2 = 0$ AND $2x - 3y + 6 = 0$ **B** $4y = 3x - 1$ AND $8y = 6x - 7$

6 WRITE THE EQUATION OF EACH CIRCLE WITH THE GIVEN CONDITIONS:

A CENTRE AT O(3, –7) AND RADIUS 3

B CENTRE AT P(3, –7) AND TANGENT TO $x + 3y - 4 = 0$

C END POINTS OF ITS DIAMETER ARE A(3, –7) AND B(4, 3)

7 FIND THE EQUATION OF THE TANGENT LINE TO THE CIRCLE WITH EQUATION $(x - 3)^2 + (y - 4)^2 = 20$ AT P(1, 0).

8 FIND THE EQUATION OF THE PARABOLA WITH THE FOLLOWING CONDITIONS.

A FOCUS AT F(–2, 0); DIRECTRIX $x = 2$

B FOCUS AT F(3, 3); VERTEX AT V(3, 2)

C VERTEX AT O(0, 0); AXIS y-AXIS; PASSES THROUGH A(−1, 1)

9 FOR EACH PARABOLA WHOSE EQUATION IS GIVEN BELOW, FIND THE FOCUS, VERTEX, DIRECTRIX AND AXIS.

A $(x-1)^2 = y+2$ **B** $x^2 = -6y$ **C** $4(x+1) = 2(y+2)^2$

10 WRITE THE EQUATION OF EACH ELLIPSE THAT SATISFIES THE FOLLOWING CONDITIONS.

A THE FOCI ARE F(3, 0) AND F′(−3, 0); VERTICES V(5, 0) AND V′(−5, 0).

B THE FOCI ARE F(3, 2) AND F′(−3, 2); THE LENGTH OF THE MAJOR AXIS IS 8.

C THE FOCI ARE F(4, 7) AND F′(−4, 7); THE LENGTH OF THE MINOR AXIS IS 9.

D THE CENTRE IS C(6, −2); ONE FOCUS IS F(3, −2) AND ONE VERTEX IS V(10, −2).

11 FIND THE FOCI AND VERTICES OF EACH OF THE ELLIPSES WHOSE EQUATIONS ARE GIVEN.

A $4x^2 + y^2 = 8$ **B** $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$

12 GIVE THE EQUATION OF A HYPERBOLA SATISFYING THE FOLLOWING CONDITIONS:

A FOCI AT F(9, 0) AND F′(−9, 0); VERTICES AT V(4, 0) AND V′(−4, 0).

B FOCI AT F(0, 6) AND F′(0, −6); LENGTH OF TRANSVERSE AXIS IS 6.

C THE FOCI AT F(0, 10) AND F′(0, −10); ASYMPTOTES $y = \pm 3x$.

13 FIND THE VERTICES, FOCI, ECCENTRICITY AND ASYMPTOTES OF THE HYPERBOLA WHOSE EQUATION IS GIVEN AND SKETCH THE HYPERBOLA.

A $9x^2 - 16y^2 = 144$ **B** $\frac{(x+3)^2}{25} - \frac{(y+1)^2}{144} = 1$

14 AN ARCH IS IN THE FORM OF A SEMI-ELLIPSE. IT IS 50 METRES WIDE AT THE BASE AND HAS A HEIGHT OF 20 METRES. HOW WIDE IS THE ARCH AT THE HEIGHT OF 10 METRES ABOVE THE BASE?

Hint:- *Take the x-axis along the base and the origin at the midpoint of the base.*

15 AN ASTRONAUT IS TO BE FIRED INTO AN ELLIPTICAL ORBIT AROUND THE EARTH HAVING A MINIMUM ALTITUDE OF 800 KM AND A MAXIMUM ALTITUDE OF 5400 KM. FIND THE EQUATION OF THE CURVE FOLLOWED BY THE ASTRONAUT. CONSIDER THE RADIUS OF THE EARTH TO BE 6400 KM.